MECHANICS

OF

INTERNAL WORK

(OR WORK OF DEFORMATION)

IN ELASTIC BODIES AND SYSTEMS IN EQUILIBRIUM

INCLUDING

THE METHOD OF LEAST WORK

BY

IRVING P. CHURCH, C.E.

ASSOC. AM. SOC. C.E.

Professor of Applied Mechanics and Hydraulics, College of Civil Engineering, Cornell University

FIRST EDITION

FIRST THOUSAND

NEW YORK

JOHN WILEY & SONS

LONDON: CHAPMAN & HALL, LIMITED

1910

THE SCIENTIFIC PRESS
ROBERT DRUMMOND AND COMPANY
BROOKLYN, N. Y.

PREFACE

Among the more modern methods used in dealing with elastic bodies in the Mechanics of Materials, those based on the properties of the derivatives of the work of deformation (or "*internal work*" as it will be called in the present volume), as first thoroughly established by Castigliano * in 1879, and including the "Method of Least Work," have been but sparingly presented in American text-books; and even in those works which include applications of these methods, such proofs of the methods themselves as may be given are generally so meagre in detail that the student is left much in the dark as to the precise conditions which must be observed to render the methods applicable.

It has therefore seemed desirable to the writer that a text-book be prepared which should deal exclusively with the "Mechanics of Internal Work in Elastic Bodies in Equilibrium" and embody special fullness of explanation and illustration, not only in the demonstrations of the methods to be employed but also in the detail of the applications; and thus indicate clearly the use, scope, and limitations of these valuable methods.

The following pages are the result of an attempt in this direction; and it is hoped that the book may prove useful not only to students of engineering, but in some degree also to members of the profession.

In this connection the attention of students who read German is directed to Müller-Breslau's standard work on "Recent Methods in

* Castigliano wrote in French, his book being entitled "*Théorie de l'Equilibre des Systèmes Elastiques.*" It was published at Turin in 1879 and has been translated into German. Prof. William Cain's paper on "*Determination of the Stresses in Elastic Systems by the Method of Least Work*" in the Transac. Am. Soc. Civ. Engs., April, 1891, is a valuable contribution to this subject; as also Prof. George F. Swain's article "*On the Application of the Principle of Virtual Velocities to the Determination of the Deflection and Stresses of Frames,*" in the Journal of the Franklin Institute for Feb., March, and April, 1883.

Strength of Materials, etc.*" Books in English by Hiroi and Martin are referred to in the present work (see p. 89); and also articles by Tilden, Hudson, and Mensch (see pp. 51, 53, and 93).

The abbreviation M. of E. will be used in referring to the writer's "Mechanics of Engineering."

CORNELL UNIVERSITY, ITHACA, N. Y.,
October 20, 1910.

* *Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen*, Leipzig, 1893 (zweite vermehrte und verbesserte Auflage).

CONTENTS

MECHANICS OF INTERNAL WORK

CHAPTER I

External and Internal Work of Elastic Structures

1. Gradual Application of Loads. The propositions or theorems of this and the two following chapters have to do with elastic bodies, or structures, undergoing slight deformations *within the elastic limit.* Certain points of the structure are supported on fixed (and inelastic) bodies, while at other points forces or loads are applied, each increasing in value gradually and simultaneously, and proportionately to the others, until their final values are reached at the same instant. During this gradual application of the external or applied forces, gradual, but *slight,* changes of form occur in the elastic body or structure itself.

2. Structure to Consist of Bars. For present purposes the body or structure will be conceived to be composed of *straight elastic bars,* each being pivoted (pin-connected) to the neighboring bars, at its two extremities (and nowhere else); all these bars lying in the same plane (plane of the paper). All external forces or loads are to be applied at joints; that is, at the pins. Each pin is perpendicular to the plane of the bars.

3. Bars Considered Elastic in Succession. In establishing the theorems, however, it will be convenient to consider that the elastic character of the body is introduced in successive stages; that is, the body will first be considered to be entirely rigid with the exception of one bar, all the remainder of the body being treated as a stiff plate or rigid body. Then, portions of this plate will be considered to be composed of elastic bars;

and so on, in successive cases, until a proposition is established for a body composed entirely of straight elastic bars.

4. Necessary Bars. Redundant Bars. If there are more bars than are necessary for the structure to hold its shape (that is, to hold it, aside from the small elastic deformation which will occur when loads are applied) it is said to contain "*redundant bars*"; though as to *which* bars are redundant may be an arbitrary matter, more or less. Thus, if a set of five bars form the four sides and one diagonal of a quadrilateral, all of these bars are necessary that the framework may hold its shape and be capable of bearing loads applied at certain joints, two other joints being supported; but if a bar be placed in the position of the other diagonal, the frame contains *one* redundant bar. Any one of the six bars could be treated as the redundant bar and the other five as the "*necessary bars*"; since, whichever one were omitted, the other five, connecting the original four vertices or joints, would constitute a frame capable of holding its shape.

m - 3 bars necessary

5. As a First General Theorem it will be proved that in a structure of elastic bars the "*work*" of the applied forces or loads is equal to the work expended on stretching or compressing the bars, all these loads being gradually applied. Or, briefly,

$$\text{External work} = \text{Internal work.} \quad . \quad . \quad . \quad . \quad (1)$$

It is restricted to cases where all the bars are accurately fitted to each other before loading and are under *no initial stress;* that is, under no stress when the application of the applied loads begins; so that all the bars begin to stretch or to shorten at the same instant and the stresses in them increase in direct proportion to their final values. Also, *there must be no change of temperature* during the application of the applied forces.

In the case of a bar under compressive stress, though designed by a "long column" formula, its amount of shortening is supposed to be proportional to the stress at any instant.

Case I. The Structure Contains only One Elastic Bar. Fig. 1 represents a structure in which the plate N is rigid while the bar hb is elastic. M is a fixed rigid support, the

straight bar hb being "pin-connected" both to N and M. A load P is applied to the plate at the point e. Originally, at the instant of first application of the load, the point b was in position a, e was at d, and the bar hb was under no stress. As a result of the gradual* application of the load P the bar stretches and finally its extremity b reaches the position of the figure and the load (force) P reaches its final full value P. At this final instant there is a certain tensile stress in the elastic bar which we shall call T. There is also a reaction, R, at the pin m, where the rigid plate is pivoted to the fixed support M. One extremity, h, of the elastic bar is also pivoted to M. We have, therefore, the rigid body N under the action of the three

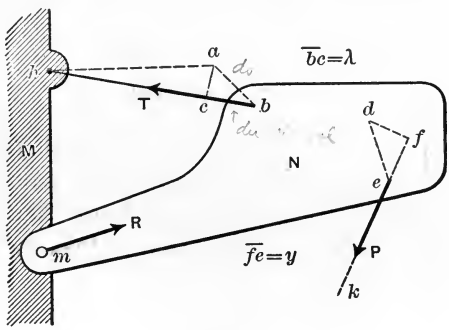

Fig. 1.

forces P, T, and R, and *in equilibrium*. During the gradual application of the load its point of application has moved from d to its final position e.

Let a perpendicular now be let fall from d upon the (present) line of action, $f \ldots k$, of the force P. The distance fe is then the projection of de upon the line $f \ldots k$. Similarly, let fall the perpendicular ac from point a upon line bh; bc, the projection of ab upon line $b \ldots h$, being thus obtained.

Since the plate N is a rigid body in equilibrium under the three forces T, P, and R, let us now apply to it the principle of *Virtual Velocities* (M. of E., p. 68) conceiving it to execute a small displacement of precisely such character and extent as

* If the force or load P is due to gravity, this gradual application may be supposed to be brought about by the progressive pouring of sand into a pail, for instance.

to bring points b and e *to their original positions* a and d, respectively; that is, a small rotation about point m. It should be remembered that this small displacement is purely imaginary, and that while it is taking place the values of the three forces P, T, and R, remain constant. For this displacement the "virtual velocity" of T is $\overline{bc}$ and occurs on the positive side of the point b [that is, on the side toward which the force T (arrow) points], while the "virtual velocity" $\overline{ef}$ of the force P falls on the negative side for that force (on the side opposite from the direction in which P points); the virtual velocity of force R being zero. Giving, therefore, the proper sign to each "virtual moment," or product, with zero as right-hand member of the equation, we have [from the $\Sigma(Pdu)=0$ of p. 68, M. of E.],

$$T\,.\,\overline{cb}-P\,.\,\overline{ef}+R\times 0=0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But evidently the distance $\overline{cb}$ is not only the virtual velocity of the force T, but also the difference (practically such), λ, between its length, $\overline{hb}$, under stress and its original length, $\overline{ha}$, when under no stress. That is, λ is the elongation of the bar. Hence we may write

$$Py=T\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which y, or $\overline{ef}$, is the "*displacement*" of the point d *in the direction of the force* P (i.e., the *projection* of $\overline{de}$ on line of force P).

6. External Work; and Internal Work. While the last result is true as it stands, as a result of the "principle of virtual velocities" applied to a rigid body at rest (the displacement mentioned being only imagined to take place), if we divide both members by 2 we obtain the form

$$\frac{P}{2}y=\frac{T}{2}\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

to be interpreted as follows: $P/2$ is the average value of the force P during the gradual movement of point d to its final position e (this force having increased from zero to P), while $T/2$, similarly, is the average stress in the elastic bar during its gradual elongation (the initial stress having been zero). Also, y is the projection upon the line of P of the "path," *de*,

of its point of application; and similarly $\overline{cb}$, or the elongation λ, is the projection, upon the line of the force T, of the "path," ab, of the point of application of this latter force. Consequently the product $\frac{P}{2}y$ is called the *work done* by the force P during its gradual application (and this is called "external work"; and if other loads were acting we should add a similar term for each one); while the product $\frac{T}{2}\lambda$ is called the "*work of deformation,*" or work done in changing the length of the elastic bar from a condition of no stress to its final condition (and the sum of a number of similar terms for a number of bars forming an elastic structure will be called the "**internal work**").

7. Case II. The Structure Contains Four Elastic Bars. Fig. 2 shows a structure consisting of a rigid plate N, and four elastic bars, namely, 1, 2, 3, and 4, connecting the plate

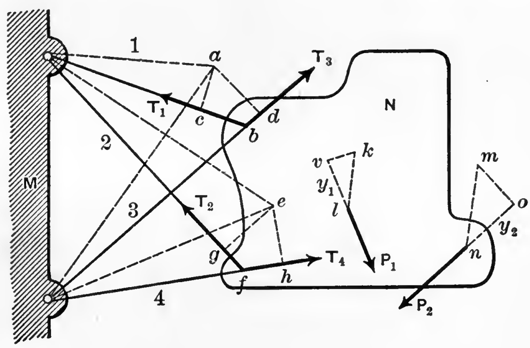

FIG. 2.

to a fixed rigid body M. To the rigid plate are applied two loads P_1 and P_2 acting (in this final position of the plate) at points l and n. When the loads were first applied these two points were at k and m, respectively, and the values of the loads were each zero. The pivots b and f where the elastic bars are connected were originally at points a and e, these points forming with points k and m the same rigid configuration which they form in their final positions (the plate N being rigid). The four bars comprehend one *redundant* bar, since

three alone would have been sufficient to enable the whole structure to hold its shape and its connection with body M; but the proposition would be equally true on the supposition that only three bars were present. During the gradual application of the two loads varying simultaneously in value from zero up (and proportionally) to their final amounts, P_1 and P_2, the point a has moved to some new position b and, likewise, e to f, k to l, and m to n; while the stresses in the four bars, whatever their initial values, attain, respectively, certain final values, T_1, T_2, T_3, and T_4. The figure, then, shows the rigid body N in this final state of equilibrium and acted on by the six forces, viz., T_1, T_2, T_3, T_4, P_1 and P_2; and to this body we shall apply the principle of virtual velocities; regarding, as the small displacement, which is imagined to occur, the very movement which would bring the four points b, f, l, and n, back to their original positions, a, e, k, and m. If from a we draw the perpendicular ad upon the line of bar 3 (prolonged), thus obtaining the point d, and also a perpendicular from a upon the line of bar 1, obtaining the point c, we note that $\overline{bd}$ is the virtual velocity of T_3, and also the shortening, λ_3, of bar 3; while $\overline{bc}$ is the virtual velocity of T_1 and also the amount of elongation, λ_1, of bar 1. Similarly, drawing at the point e the perpendiculars eh and eg we determine $\overline{fh}$, which is at the same time the virtual velocity of T_4 and also the shortening λ_4 of bar 4; and $\overline{fg}$ as the virtual velocity of T_2 and the elongation λ_2 of bar 2. At each of the points k and m, dropping perpendiculars upon the lines of action of P_1 and P_2, respectively, we determine the virtual velocity $\overline{vl}(=y_1)$ of P_1 and also that, $\overline{on}$, or y_2, of P_2. Forming, then, the algebraic sum of the products of each of the six forces by its virtual velocity, representing the respective elongations or shortenings of the bars by λ_1, λ_2, etc., and writing the sum equal to zero, we obtain:

$$T_1\lambda_1+T_2\lambda_2+T_3\lambda_3+T_4\lambda_4-P_1y_1-P_2y_2=0, \quad . \quad . \quad (5)$$

or, by transposition,

$$P_1y_1+P_2y_2=T_1\lambda_1+T_2\lambda_2+T_3\lambda_3+T_4\lambda_4. \quad . \quad . \quad . \quad (6)$$

At this stage of the present case there is no object in dividing each term of this equation by 2 and calling each term a certain

amount of work done by, or upon, each force during the gradual application of the loads P_1 and P_2; since it may have been necessary, before this application, in putting the bars together, to strain one of the bars (the last one fitted; that is, the "redundant" bar) into the position over the pivots at its extremities; which operation would create in all the four bars certain initial stresses, different from the values which the stresses in these bars will have attained at the end of the gradual application of the loads; therefore, the mean value of the stress in bar 1 (for instance) during the gradual application would *not* be one-half of the final value, T_1. Hence eq. (6) will be allowed to stand in its present form, as a mere consequence of the principle of virtual velocities, without bringing into play any idea of the "work done" by any force during the gradual application of the loads.

8. Special Case of the Preceding. No Initial Stress. Let us suppose that although there is one redundant bar, the last bar fitted is of just the proper length to fit over the pivots provided for its extremities *without strain;* there will then be **no initial stress** in the bars; and when the gradual application of the loads begins, all the bars will begin simultaneously either to lengthen or to shorten, and the stress in each will change gradually from its initial value zero to its final value; so that the mean value is one-half the final. In such a case *all the terms in the right-hand member of eq.* (6) *are positive,* whether a bar is subjected to a lengthening, or to a shortening. Bar 1, for instance, suffers an elongation, so that the force T_1 points to the left and the virtual velocity $\overline{cb}$ falls on the same side of the point b as that in which the arrow (or force) T_1 is directed, so that the product $T_1\lambda_1$ is positive in eq. (5). Now considering the case of bar 4, we note from the figure that it is subjected to a shortening, and hence to a compressive force, T_4. This force, as an action on body N, must therefore point to the right in the figure; where we also note that the virtual velocity for this force, namely, hf, falls on the same side of f as the arrow denoting the force; so that here again the product (viz., $T_4\lambda_4$) will be positive in eq. (5); which is the same result that we reached for the bar in tension. From this it follows, then, that all the products of the form $T\lambda$ in the right-hand

member of eq. (6), in this case of *no initial stress* in the bars, are *positive;* from which fact a very important relation will result, in future propositions. As to the left-hand member of eq. (6), the products of the form Py may be positive, or may be negative; according to the positions of the respective applied forces, or loads, P_1, P_2, etc., since the movement of the rigid body N during the gradual application of these loads may be such, and the forces may be so directed, that the projections, or virtual velocities, y_1, y_2, etc., may, or may not, fall on the same side of l (or n, respectively) as the arrow showing the direction of the force.

9. No Redundant Bars; then All Products Tλ Positive in eq. (6). If in Case II there has been no redundant bars connecting the rigid bodies M and N (in other words, if no more than the "necessary" bars had been used) there would be *no initial stresses* in the bars; so that, as the loads are gradually applied, each stress increases gradually from a zero value to its final, and each term on the right in eq. (6) is *positive,* as already shown for the case of no initial stresses with redundant bars.

It should therefore be noted, that in both cases, viz., of redundant bars, and of simply necessary bars, the average stress is not equal to one-half the final *unless the initial stresses are each zero* (which is always so for the case of simply necessary bars, but not necessarily true in a redundant system).

10. Case III. The Structure Contains Five Elastic Bars, No One of which is Redundant. (N.B. The absence of a redundant bar does not vitiate the truth of the proposition. One or more such bars might be introduced without changing the nature of the demonstration; the object of omitting them is to reduce the amount of detail. See § 14.)

Here, as before, we have (see Fig. 3) a rigid plate N connected to a rigid support M by a number of elastic bars, so arranged as to be five in number and to involve no redundant bars. Of the five joints, or pivots, having to do with these bars, p and q remain stationary during the application of the loads P_1 and P_2, while the other three, now at b, f, and s, were originally at a, e, and r, respectively, when the gradual application of the loads P_1 and P_2 began. At this initial instant each of the five bars was under a *zero* stress.

11. It will be noticed from the figure that the only bars the stresses in which act on points of the rigid body N are bars 3, 4, and 5, and that in the special case shown in this figure bar 3 has been *shortened,* so that its action against body N is a force T_3 pointing toward the right. As to whether bars 4 and 5 are in tension or in compression does not directly appear in the figure since the movement of joint b would not be thoroughly known except in some particular numerical case. As a supposition to be consistently followed when joint b comes up for consideration, we shall suppose that the action of bar 4 upon body N is a tension, and hence directed upward at point f;

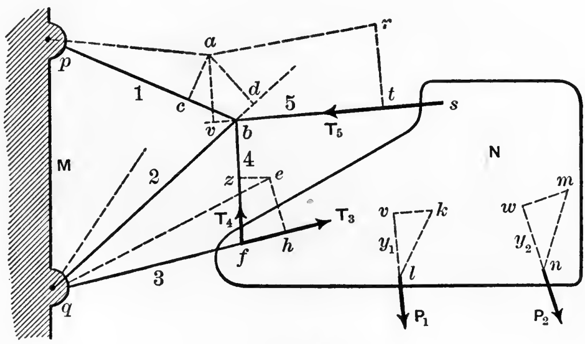

Fig. 3.

and that the action of bar 5 upon body N is also a tension, and hence directed to the left at point s.

12. As before, we shall apply the principle of virtual velocities to the rigid body N in equilibrium under the five forces T_3, T_4, T_5, P_1, and P_2; imagining by way of a small displacement that the points of application (s, f, l, and n) shift *back to their original positions* (r, e, k, and m) and letting fall from each of these last points, respectively, the proper perpendiculars upon the lines of action of the respective forces. The five forces concerned remain constant during this shift. In this way we obtain the virtual velocity $\overline{ts}$ of T_5; $\overline{fh}$, that of T_3; $\overline{zf}$, that of T_4; $\overline{vl}$, that of P_1; and $\overline{wn}$, that of P_2. As before, $\overline{fh}$, being the elongation of bar 3, will be called λ_3; also $\overline{vl}$, y; and $\overline{wn}$, y_2; whereas, for the present, $\overline{st}$ and $\overline{zf}$, as quantities

(lengths), will be written $\overline{st}$ and $\overline{zf}$. Forming the various products, then, or "virtual moments," we have

$$T_3\lambda_3 + T_4 \,.\, \overline{zf} + T_5 \,.\, \overline{st} - P_1y_1 - P_2y_2 = 0. \quad . \quad . \quad (7)$$

If now we consider the joint or pivot at b as a separate "free body," we find the system of concurrent forces, shown in Fig. 4, acting on the pivot; and there have been reproduced in this figure, in their proper relative positions, the point a, the four perpendiculars ac, av, ad, and au, as also the prolongations bu and bv of the two bars 4 and 2. In other words, we have in this figure the respective virtual velocities $\overline{bc}$, $\overline{bd}$, $\overline{bu}$, and $\overline{bv}$, of the four stresses, or forces, acting on the pin b.

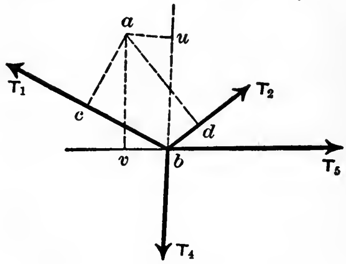

FIG. 4.

Now conceiving, as a basis for applying "virtual velocities," that the pin b is displaced through the small distance $\overline{ba}$, and forming the proper products, we derive

$$T_1 \,.\, \overline{bc} + T_2 \,.\, \overline{bd} - T_4 \,.\, \overline{ub} - T_5 \,.\, \overline{vb} = 0. \quad . \quad . \quad . \quad (8)$$

It is now to be noted, that the projection cb is equal to the elongation, λ_1, or bar 1, and similarly, that the projection db is the elongation, λ_2, or bar 2, so that eq. (8) may be rewritten

$$T_1\lambda_1 + T_2\lambda_2 - T_4 \,.\, \overline{ub} - T_5 \,.\, \overline{vb} = 0. \quad . \quad . \quad . \quad . \quad (9)$$

We note again, examining both Fig. 3 and Fig. 4 that

$$\overline{zf} - \overline{ub} = \lambda_4 = \text{the elongation of bar 4;}$$

and $$\overline{st} - \overline{vb} = \lambda_5 = \text{the elongation of bar 5.}$$

These substitutions having been made in the equation resulting from the adding of (7) and (9) we obtain, after transposition,

$$P_1y_1 + P_2y_2 = T_1\lambda_1 + T_2\lambda_2 + T_3\lambda_3 + T_4\lambda_4 + T_5\lambda_5. \quad . \quad (10)$$

13. Attention is now again called to the fact that each term of the form $T\lambda$ in the right-hand member of eq. (10) is *positive,* and this feature *will always occur,* in systems having no redundant bars, for reasons previously stated (§ 8); whereas,

in the left-hand member, where we have to do only with the applied forces and the respective projections y_1, etc., some of the terms may be negative; in accordance with the special conditions of any particular case.

14. In case there had been **redundant bars in this case,** eq. (10) would have resulted in the general form as already derived; but if the idea of work were to be introduced, we should have to remember that at the beginning of the gradual application of the loads the various elastic bars might be already under stress (that is, there might be "initial stresses"); so that $\frac{T_1}{2}, \frac{T_2}{2}$, etc., would not necessarily be the average values of the stresses in bar 1, bar 2, etc., respectively, during the gradual application of the loads. Also the terms of the form $T\lambda$ in the right-hand member of eq. (10) would not all necessarily be positive (though such would be the case if the bars were so accurately fitted originally that there are no initial stresses).

15. It now becomes evident from the nature of the steps already taken, that eq. (10) is general for any structure of elastic bars which become slightly changed in length by reason of the gradual application of loads P_1, etc., and that it is true whether (there being redundant bars) there were already stresses ("initial stresses") existing in some or all of the bars, or not; so long as each symbol T denotes the stress ~~the stress~~ in a bar at the instant when all the loading is in place and the body at rest, and λ denotes the change of length occurring in each bar as due to the gradual loading of the structure (and *not* the difference between its final length and that which it would have under no stress whatever.

16. Limitation to Zero Initial Stresses. λ in Terms of T. We are now going to limit, very specifically, the conditions of our problems by stipulating that, before the structure is loaded, all redundant bars are of *just the proper length*, respectively, to fit over their pivots without stress of either kind, tension or compression. *Under these circumstances*, the stress in each bar is zero before loading takes place, and the quantity λ denotes the difference between the length of a bar under its final stress T and its length under no stress. That is, λ is the *elongation*

due to T and *is proportional to* T, bearing to it the relation (see p. 203, M. of E.),

$$\lambda=\frac{Tl}{FE}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

where F is the sectional area (square inches, say) of the bar (which is understood to be a *prism* in form, and *homogeneous* in material), l is the length, and E is the modulus of elasticity (pounds per square inch, for instance) of the material.

Hence, *under these special conditions*, eq. (10), after division of both members by 2, may be rewritten in the form,

$$\frac{P_1y_1}{2}+\frac{P_2y_2}{2}+\text{etc.}=\frac{T_1\lambda_1}{2}+\frac{T_2\lambda_2}{2}+\text{etc.}, \quad . \quad . \quad . \qquad (12)$$

or [see eq. (11)],

$$\tfrac{1}{2}P_1y_1+\tfrac{1}{2}P_2y_2+\ldots=\tfrac{1}{2}\cdot\frac{l_1}{F_1E_1}T_1{}^2+\tfrac{1}{2}\cdot\frac{l_2}{F_2E_2}\cdot T_2{}^2+\ldots, \qquad (13)$$

the form in eq. (13) being of such a nature as to embody certain **convenient conceptions,** as follows:

17. "External Work." If we assume that the external forces, or "loads," are applied to the elastic structure simultaneously, each increasing very gradually but in proportion to the others, from a value of zero to its full (or "final") value P_1 (or P_2, etc.) (this could be done practically by the progressive pouring of sand into pails, for instance); their points of application will be deflected or displaced through the various distances y_1, y_2, etc., *measured in the direction* of the corresponding load or "external force"; and the product of $\frac{1}{2}P_1$ (which is the average value of load No. 1 during this motion) by the deflection y_1 (*in direction of force*) is called the "work" done by this force during the gradual loading of the structure. Similarly, $\frac{1}{2}P_2y_2$ is the work done by load No. 2, etc. (Some of *these* products may be negative.) Hence the left-hand member of eq. (13) may be called the "External Work," or work of the applied loads.

18. Internal Work Equals External Work. During this gradual and progressive application of the external forces, tensile and compressive stresses are created in all the elastic bars, the stress in each bar increasing gradually, and simul-

taneously with those in all the others, from a value of zero to its full "final value" T (pounds, for instance) which it reaches at the same instant that the stresses in the other bars reach their respective "final" stresses. In this progressive increase of stress, the stress in any bar is always directly proportional to the change of length, and hence the average stress in any bar (for purpose of formulating the work done in producing that change of length) is one-half of its final value T, and the product of average stress by elongation, that is, $\frac{T}{2}\lambda$, is called the work done in changing the length of the bar. Now the right-hand member of eq. (12) is seen to consist of the sum of all such products for all the bars of the structure and is called the **"Internal Work"** of the structure or system of elastic bars, and will be denoted by the symbol U; that is [see also eq. (13)], U denotes

$$\tfrac{1}{2} \cdot \frac{l_1}{F_1E_1} \cdot T_1{}^2 + \tfrac{1}{2} \cdot \frac{l_2}{F_2E_2} T_2{}^2 + \ldots, \text{etc.}; \quad . \quad . \quad (14)$$

or,

$$U = \Sigma\left(\tfrac{1}{2} \cdot \frac{l}{FE} \cdot T^2\right); \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and is also called the "*work of (elastic) deformation.*" Eq. (12) or (13) can therefore be read,

External Work = Internal Work. . . . (16)

19. Note. It must be repeated that in case there are redundant bars these must be of just such original lengths as to fit into their places without stress before the structure is loaded; or, to express it in another way, the "necessary bars" having been assembled so as to give to the structure its proper form, these necessary bars being without strain since no loads are as yet applied, the other bars, viz., the redundant bars, are supposed to be of just the proper lengths to fit upon their various pivots without subjecting themselves or the necessary bars to any stress. Hence when the loads are now gradually applied to the structure, the initial value of the stress in each bar is zero. It must also be understood *that no change of temperature occurs* in the structure during the application of the loads.

20. The arbitrary phrases just adopted (*internal* and *external work*, etc.) are convenient, but not really necessary, since eqs. (10) and (12) are mathematically correct for the quantities indicated by the respective symbols, whether any names are given to the products in question, or not; and it is really immaterial (so far as the final result is concerned), in conceiving the loads to be placed on the structure, whether they are applied in the manner indicated (simultaneously) or whether they are applied in succession, each gradually; the final result is the same.

For brevity, the fraction $\frac{l_1}{F_1 E_1}$, which is dependent only on the dimensions and the material of bar No. 1, may be called C_1 (and similarly for the other bars we write C_2, C_3, C_4, etc.; respectively), and hence eq. (14) may be written

$$U = \textit{internal work}, = \tfrac{1}{2}C_1 T_1^2 + \tfrac{1}{2}C_2 T_2^2 + \text{etc.}; \qquad (17)$$

or,

$$U = \Sigma(\tfrac{1}{2}CT^2); \qquad (17a)$$

while the expression for

$$\textit{external work} \text{ is } \quad \tfrac{1}{2}P_1 y_1 + \tfrac{1}{2}P_2 y_2 + \text{etc.} \qquad (18)$$

Note *that each term in* (17) *is positive;* while in (18) some terms may be negative.

CHAPTER II

THE DERIVATIVES OF INTERNAL WORK

21. Castigliano's Theorem. Derivative of Internal Work with Respect to an External Force, or "Load." Statement: *In any structure of elastic bars, at rest (under no initial stresses, with or without redundant bars;* supported by fixed points, or by fixed and smooth supporting surfaces, or both) the "derivative" (or "first differential coefficient") of the total internal work with respect to any one of the external forces* (P_1, for instance) *is equal to the displacement* (y_1) *of the point of application of that force, in the direction of the force* (that is, the *projection*, upon the action line of the force, of the actual displacement of the point of application).

That is, we are to prove that

$$\frac{dU}{dP_1}=y_1. \qquad (19)$$

In the next figure, Fig. 5, the network of elastic bars of which the structure is really composed is not shown. The continuous

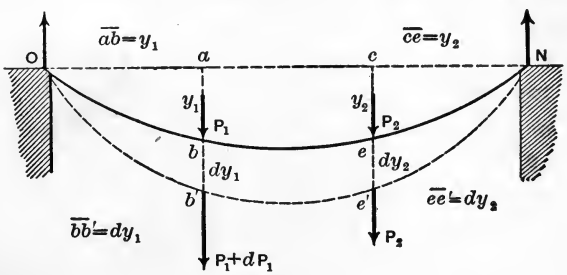

FIG. 5.

straight line *OacN* represents the structure (for example, a bridge truss) under no load whatever, while the curved continuous line *ObeN* represents the same structure after the gradual application of the two (vertical) loads P_1 and P_2. O and N represent two fixed supports. The point O of the structure is fixed, but the point (or joint) N bears against a *smooth* surface

* Note the restriction stated in § 19.

of the fixed support at that end. During the application of the two loads the points of application have moved to final positions b and e respectively, so that their total (vertical) displacements at this stage are $\overline{ab}$, or y_1, and $\overline{ce}$, or y_2, respectively. $ObeN$ will be the new position of the truss, and the system of external forces acting upon the truss consists of the two loads P_1 and P_2 and the two reactions P_0 and P_n. Whatever the value of the internal work may be at this stage, we know it to be equal to the external work.

Note. It is to be noted that the external work done by the reaction P_0 is zero, because point O does not move; and that the work done by P_n is zero, because its point of application moves *at right angles* to the action line of P_n, and hence the projection of its displacement upon the action line of P_n is zero. The reaction of a *smooth* surface is necessarily in a line normal to that surface. We may therefore write

$$U=\frac{P_1y_1}{2}+\frac{P_2y_2}{2}+\frac{P_0\times 0}{2}+\frac{P_n\times 0}{2}; \quad . \quad . \quad . \quad . \quad (19a)$$

that is

$$U=\tfrac{1}{2}P_1y_1+\tfrac{1}{2}P_2y_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

We shall now note the effect, in producing an increment, dU, in the value of U, of giving to *one* of the external forces (say, P_1) an increment, dP_1. This increase in P_1 will cause the truss to descend slightly to a lower position (dotted) $Ob'e'N$; creating changes dy_1 and dy_2 in *both* the displacements y_1 and y_2. (P_2 remains of the same value as before.) This will cause a small addition or increment,

$$\frac{P_1+(P_1+dP_1)}{2}dy_1+P_2dy_2,$$

in the external work, and consequently an equal increment (dU) in the internal work; whence, neglecting the term involving the product of two differentials, we have

$$dU=P_1dy_1+P_2dy_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

But we can express dU in another form, as follows: We first conceive the truss to be restored to the unstrained position $OacN$ (by removing all loads), and then consider the pressures (or loads) at a and c to have (P_1+dP_1) and P_2 as their respective

final values, and to be applied gradually and progressively (from zero), while always maintaining to each other the constant ratio subsisting between their final values. When these final values are reached, we find the structure in the position (dotted) $Ob'e'N$; and the whole external work done in this operation of "*deformation*" (that is, from state $OacN$ to state $Ob'e'N$) is

$$U' = \tfrac{1}{2}(P_1 + dP_1)(y_1 + dy_1) + \tfrac{1}{2}P_2(y_2 + dy_2); \quad . \quad . \quad . \quad (22)$$

i.e., neglecting the term involving the product of two differentials, and rearranging terms,

$$U' = [\tfrac{1}{2}P_1y_1 + \tfrac{1}{2}P_2y_2] + \tfrac{1}{2}(dP_1)y_1 + \tfrac{1}{2}P_1dy_1 + \tfrac{1}{2}P_2dy_2. \qquad (23)$$

Now the difference, $U' - U$, must be equal to dU, the increment accruing to the value of U as occasioned by giving to P_1 an increment dP_1; hence, from eqs. (20) and (23),

$$dU = \tfrac{1}{2}(dP_1)y_1 + \tfrac{1}{2}P_1dy_1 + \tfrac{1}{2}P_2dy_2. \quad . \quad . \quad . \quad (24)$$

But, from eq. (21), $\tfrac{1}{2}P_1dy_1 + \tfrac{1}{2}P_2dy_2 = \tfrac{1}{2}dU$; and hence eq. (24) becomes

$$dU = (dP_1)y_1; \quad \text{or,} \quad \frac{\mathbf{dU}}{\mathbf{dP_1}} = \mathbf{y_1}, \quad . \quad . \quad . \quad . \quad (25)$$

that is, the derivative of U (total internal work) with respect to any one load, or external force, P, is equal to the displacement (y) in the direction of this P (i.e., the projection of the actual displacement upon the line of P) of the point of application of P.

This is the fundamental theorem or principle in the subject of internal and external work in elastic structures, Castigliano's Theorem.

If therefore the total internal work, $U = \Sigma\left(\dfrac{lT^2}{2FE}\right)$, or $U = \Sigma(\tfrac{1}{2}CT^2)$, of the elastic structure can be expressed as a function of one of the external forces (say, P_1), we may find the displacement y_1 of the point of application of that load, *in the direction of the force*, by differentiating the quantity U with respect to P_1.

Although in the foregoing there has been, besides P_1, only one other load (P_2) [which retains a fixed value during the demonstration] the result would evidently have been the same

in case there had been any number of other loads besides P_1. (P_2 and other similar loads are independent of the value of P_1, so that the derivative of one of them with respect to P_1 would be zero). The reactions P_0 and P_n of Fig. 5 are, of course, dependent on P_1; so that $\frac{dP_0}{dP_1}$, for instance, would not be zero.

It is to be particularly noted in the foregoing demonstration that the mode of support is of such character that the reaction at a supporting point is that of a *fixed point* or is offered by a *smooth and unyielding surface. A rough supporting surface, along which slipping may occur, is precluded.* An external force doing zero work may be called a **" neutral " force.**

Although for simplicity the loads or external forces P_1, P_2, etc., have been taken in a vertical position any oblique position might have been assumed for them without altering the rigor of the demonstration; nor need the line ON of the structure be taken in a horizontal position necessarily. Also it is understood that the support furnished by the supporting bodies is completely effective whatever change of value may occur in the single load which has been selected as a variable.

21a. Theorem for Variable Displacement. Castigliano has proved a second theorem which asserts that if the displacement (y) of the point of application of a load (P) (in its own direction) be made to vary, the displacements of all the other loads remaining fixed in value, then,

$$\frac{dU}{dy}=P;$$

but this theorem is of little practical utility and has not come into prominence.

22. Another Proof (Partly Geometrical) of Castigliano's Theorem. In Fig. 6 have been reproduced the various positions a, c, b, e, etc., of the points of application of the two external forces or loads, P_1 and P_2, of Fig. 5, the values of the displacements y_1 and y_2 and their increments dy_1 and dy_2 being indicated. Before P_1 received its increment dP_1, the work done by the two forces, each increasing gradually from zero and in proportion to final values P_1 and P_2, is represented

in Fig. 6 by the two shaded triangles *abh* and *cem*, *bh* being perpendicular to *ab* and made equal (by scale) to the final value P_1. Similarly (on same scale) *em* is equal to P_2. The areas of these triangles, therefore, are respectively the work items $\frac{1}{2}P_1y_1$ and $\frac{1}{2}P_2y_2$. If now P_1 receive its increment, dP_1 (a little more sand being poured into that pail), while P_2 remains unchanged, the additional work done at *b* will be represented by the (shaded) trapezoid *bhkb′*, and that at *e* by the (shaded) rectangle *emne′*; *ik* being equal to dP_1. In this first mode of loading the structure, then (the loads being $(P+dP_1)$ and P_2), the external work is represented by the shaded areas of Fig. 6.

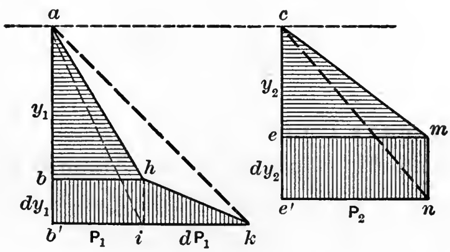

Fig. 6.

Let now the structure be unloaded (pails emptied), and then reloaded; care being taken, in this reloading, to make the amount of force (amount of sand in pails), at each instant during the whole process, proportional to P_1+dP_1, and P_2, as final values; instead of P_1 and P_2. In this second mode of loading, with same final values P_1+dP_1 and P_2 as in the first, the total external work done is represented by the two areas of the two triangles *akb′* and *cne′*. The former area is seen to be *in excess* of the shaded parts in the left of the figure, by the area of triangle *ahk*; while the latter *is less* than the shaded part in the right of the figure, by the area of triangle *cmn*. Now this excess (*ahk*) on the left must be equal to the deficiency (*cnm*) on the right; since, whichever of the two modes of loading be conceived to take place, the final state of the component bars, and hence the amount of internal work, U', are the same in the two cases; and consequently the amounts of the total external work are equal in the two cases. That is, *area of*

triangle ahk = area of triangle cnm. Now area *ahk* = +triangle *aik* − triangle *aih* − triangle *hik*; i.e.,

$$\overline{ahk} = \tfrac{1}{2}dP_1[y_1 + dy_1] - \tfrac{1}{2}P_1 \,.\, dy_1 - \tfrac{1}{2}(dP_1)dy_1, \quad . \quad (25a)$$

and

$$\overline{cnm} = \tfrac{1}{2}P_2 \,.\, dy_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25b)$$

Equating (25*a*) and (25*b*), we have, neglecting products of two differentials,

$$\tfrac{1}{2}(dP_1)y_1 = \tfrac{1}{2}P_1dy_1 + \tfrac{1}{2}P_2dy_2. \quad . \quad . \quad . \quad . \quad (25c)$$

But, by eq. (21), the right-hand member of eq. (25*c*) is equal to $\frac{1}{2}dU$; and hence, finally,

$$\frac{dU}{dP_1} = y_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which is *Castigliano's Theorem.*

23. Example of Application of Castigliano's Theorem. The structure in Fig. 7 contains seven bars, three horizontal and four at 45°. Pins *o* and *n* rest on the *smooth* horizontal

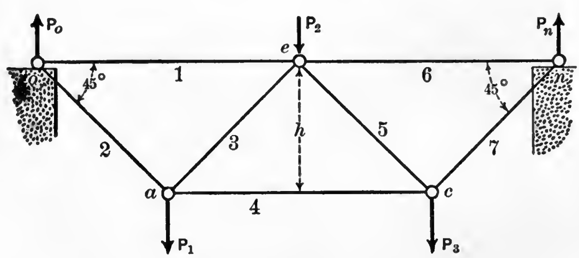

Fig. 7.

surfaces of two fixed piers, so that the reactions P_0 and P_n are vertical. The three vertical forces (loads) P_1, P_2, and P_3 are not equal to each other, necessarily.

It is required to find the *vertical* displacement of joint *a*, where the load P_1 is applied (i.e., the *projection*, upon line of P_1, of movement of *a*). (As to whether *a* moves horizontally, and to what extent, depends on the horizontal sliding of one or both of the pins at *o* and *n*.

There being *no redundant bars,* after finding the values of P_0 and P_n by ordinary statics, viz.:

$$P_0=\tfrac{3}{4}P_1+\frac{P_2}{2}+\frac{P_3}{4}; \quad \text{and} \quad P_n=\frac{P_1}{4}+\frac{P_2}{2}+\tfrac{3}{4}P_3; \quad . \quad (26)$$

we find by ordinary statics the values of the seven stresses in the respective bars (in terms of the given loads and the reactions). As to whether any such stress is tensile or compressive, this is immaterial; since the internal work due to each one is of necessity positive (see § 9). The following values are obtained:

$$\left.\begin{array}{lll} T_1=P_0; & T_2=P_0\sqrt{2}; & T_3=(P_0-P_1)\sqrt{2}; \\ T_6=P_n; & T_7=P_n\sqrt{2}; & T_5=(P_n-P_3)\sqrt{2}; \\ \text{and} & T_4=2P_0-P_1. & \end{array}\right\} \quad . \quad (27)$$

Since we now are to consider P_1 as variable each T must be expressed as a function of the variable P_1 and of the constants P_2 and P_3. This is done by the use of eqs. (26); and the derivative of each T with respect to P_1 is then easily obtained and set down in the proper column. In this way the following values are obtained:

$$\left.\begin{array}{lll} T_1=\tfrac{3}{4}P_1+\dfrac{P_2}{2}+\dfrac{P_3}{4}; & \therefore & \dfrac{dT_1}{dP_1}=\dfrac{3}{4}; \\ T_2=\sqrt{2}\left[\tfrac{3}{4}P_1+\dfrac{P_2}{2}+\dfrac{P_3}{4}\right]; & \text{and} & \dfrac{dT_2}{dP_1}=\dfrac{3\sqrt{2}}{4}; \\ T_3=\sqrt{2}\left[\dfrac{P_2}{2}-\dfrac{P_1}{4}+\dfrac{P_3}{4}\right]; & `` & \dfrac{dT_3}{dP_1}=-\dfrac{\sqrt{2}}{4}; \\ T_4=\dfrac{P_1}{2}+P_2+\dfrac{P_3}{2}; & `` & \dfrac{dT_4}{dP_1}=\dfrac{1}{2}; \\ T_5=\sqrt{2}\left[\dfrac{P_1}{4}+\dfrac{P_2}{2}-\dfrac{P_3}{4}\right]; & `` & \dfrac{dT_5}{dP_1}=\dfrac{\sqrt{2}}{4}; \\ T_6=\dfrac{P_1}{4}+\dfrac{P_2}{2}+\dfrac{3}{4}P_3; & `` & \dfrac{dT_6}{dP_1}=\dfrac{1}{4}; \\ T_7=\sqrt{2}\left[\dfrac{P_1}{4}+\dfrac{P_2}{2}+\dfrac{3}{4}P_3\right]; & `` & \dfrac{dT_7}{dP_1}=\dfrac{\sqrt{2}}{4}. \end{array}\right\} \quad (28)$$

Before we apply eq. (25) (Castigliano's Theorem) to finding the vertical displacement y_1 of joint a we should note that, in general [see eqs. (14), (17), (17a)], since

$$U=\tfrac{1}{2}C_1T_1^2+\tfrac{1}{2}C_2T_2^2+\ldots, \text{ etc.},$$

the derivative $\frac{dU}{dP_1}$ can be written in the form,

$$\frac{dU}{dP_1}=C_1T_1\left[\frac{dT_1}{dP_1}\right]+C_2T_2\left[\frac{dT_2}{dP_1}\right]+\ldots, \text{ etc.} \quad . \quad . \quad (29)$$

[Substitution from eq. (28) in this general form gives rise to less complication than in (17).] Hence, since $\frac{dU}{dP_1}=y_1$, we have

$$\left\{\begin{array}{l} \frac{3}{4}C_1\left(\frac{3}{4}P_1+\frac{P_2}{2}+\frac{P_3}{4}\right)+\frac{3}{2}C_2\left(\frac{3}{4}P_1+\frac{P_2}{2}+\frac{P_3}{4}\right) \\ -\frac{1}{2}C_3\left(\frac{P_2}{2}-\frac{P_1}{4}+\frac{P_3}{4}\right)+\frac{1}{2}C_4\left(\frac{P_1}{2}+P_2+\frac{P_3}{2}\right) \\ +\frac{1}{2}C_5\left(\frac{P_1}{4}+\frac{P_2}{2}-\frac{P_3}{4}\right)+\frac{1}{4}C_6\left(\frac{P_1}{4}+\frac{P_2}{2}+\frac{3}{4}P_3\right) \\ +\frac{1}{2}C_7\left(\frac{P_1}{4}+\frac{P_2}{2}+\frac{3}{4}P_3\right) \end{array}\right\}=y_1, \quad (30)$$

for the displacement of the point (or joint) a, where P_1 is applied, in a vertical direction (P_1 being vertical); as due to three loads (i.e., neglecting the weights of the bars) with this mode of support.

23a. Fraenkel's Formula. Since the three loads P_1, P_2, and P_3 may have any values so long as the elastic limit is not passed in any bar, it is evident that by making $P_1=zero$ we obtain the deflection of point a as due to the loads P_2 and P_3 only; which leads to the following convenient rule: *To find the displacement, in a given direction, of a joint* (of a loaded truss or elastic structure) *where there is no load applied, conceive a load or force, P_1, applied at this joint, in the given direction, form an expression for y_1, by the same procedure as shown in foregoing example; then make $P_1=zero$ (Fraenkel's Formula).* N.B. If any of the "loads," or forces, acting on the truss in Fig. 7 were not vertical, either pin o or n would have to be fixed.

24. Numerical Case of Foregoing Example (in § 23). Let the lengths of the bars and the amounts of the loads be as shown in Fig. 7*a*, viz.:

$$l_1 = l_4 = l_6 = 20 \text{ ft.} = 240 \text{ in.};$$
$$l_2 = l_3 = l_5 = l_7 = 14.14 \text{ ft.} = 169.7 \text{ in.};$$
$$P_1 = 8 \text{ tons}, \quad P_2 = 2 \text{ tons}, \quad \text{and} \quad P_3 = 4 \text{ tons}.$$

Let the sectional area of each bar be $F_1 = F_2 = F_3$, etc., $= 2$ sq.in., and the value of E the same for all, 30,000,000 lbs. per sq.in.

We have, therefore, with the inch and *ton* as units,

$$C_1 = \frac{l_1}{F_1 E_1} = \frac{240}{2 \times 15000} = 0.008; \quad \text{also} \quad C_4 = C_6 = 0.008;$$

while

$$C_2 = \frac{169.08}{2 \times 15000} = 0.00565; \quad \therefore \ C_3 = C_5 = C_7 = 0.00565.$$

Above values in the various terms of eq. (30) give rise to

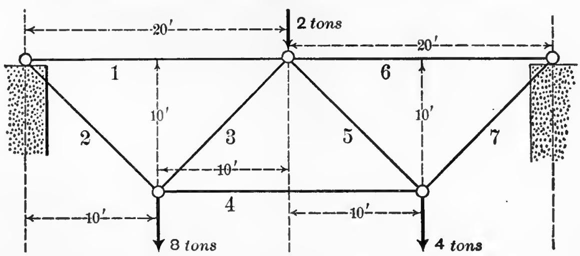

FIG. 7*a*.

$$\tfrac{3}{4}(0.008) \quad (6+1+1) = +0.0480 \text{ inches}$$
$$\tfrac{3}{2}(0.00565)(6+1+1) = +0.0678 \quad ``$$
$$-\tfrac{1}{2}(0.00565)(1-2+1) = -0.0000 \quad ``$$
$$\tfrac{1}{2}(0.008) \quad (4+2+2) = +0.0320 \quad ``$$
$$\tfrac{1}{2}(0.00565)(2+1-1) = +0.0056 \quad ``$$
$$\tfrac{1}{4}(0.008) \quad (2+1+3) = +0.0120 \quad ``$$
$$\tfrac{1}{2}(0.00565)(2+1+3) = +0.0169 \quad ``$$

and hence the vertical displacement $\Big\} \; y_1 = +0.1823$ inches.

(N.B. It should be understood that in this case the pins at the two supports are free to slide horizontally on the smooth horizontal surfaces of the *fixed* piers, where the reactions are therefore vertical in direction; and that the movements of these pins are *at right angles to the reactions* so that the external work done by the reactions is *zero*. If the piers could yield or settle during the gradual increase of the reactions (due to gradual application of loads) this would not be true (*zero* work).)

25. Derivative of Internal Work with Respect to the Moment of an External Couple. Among the external forces holding an elastic structure in equilibrium let there be a couple consisting of two equal, parallel, but oppositely directed forces, P_1 and P_2, applied at the extremities of a *rigid* cross-bar *bd*, attached to an elastic structure. Let *a* denote the perpendicular distance between P_1 and P_2 ("*arm*" of the couple).

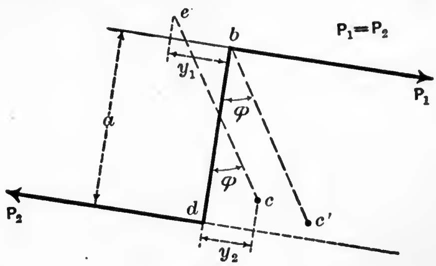

FIG. 8.

During the gradual application of the loads or external forces to the structure, the point of application of P_1 has been displaced from *e* to *b* (the projection of this displacement upon the line of P_1 may be denoted by y_1); that of P_2, from *c* to *d* (its projection on the line of P_2 being y_2). Then, by eq. (25), we have

$$\frac{dU}{dP_1}=y_1; \quad \text{and} \quad \frac{dU}{dP_2}=y_2. \quad . \quad . \quad . \quad . \quad (31)$$

The angle ϕ, between the original and the final position of the rigid cross-bar, is *so small* in any practical case that for $\tan \phi$ we may write ϕ (i.e., in "*radians*," or circular measure) and note, after drawing *bc'* parallel to *ec*, that

$$\frac{\overline{c'd}}{a}=\frac{y_1+y_2}{a}=\tan \phi; \quad . \quad . \quad . \quad . \quad . \quad (32)$$

in which for tan ϕ we now write ϕ; and, for y_1 and y_2, their values from eq. (31); whence

$$\frac{dU}{(a\,.\,dP_1)}+\frac{dU}{(a\,.\,dP_2)}=\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

But $a\,.\,dP_1=d(P_1a)$ =the differential of the moment M, $=(P_1a)$ $(=P_2a)$ of the couple; and the left-hand member of eq. (33) expresses the complete derivative of U with respect to M; and hence we may write

$$\frac{dU}{dM}=\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Evidently, from the figure, a positive value for ϕ refers to a "clockwise" displacement, or turning, of the cross-bar, if the couple itself has a clockwise moment; i.e., in general, a positive ϕ means an angular displacement of the same nature as that of the moment of the couple; and *vice versa.*

For instance, a value of 0.0174 for ϕ means an angle of 1°.

CHAPTER III

The "Theorem of Least Work"; with Applications to Systems of Bars in Equilibrium under Loads

26. Statically Indeterminate Structures. A loaded structure containing one or more redundant (or "unnecessary") bars is called "*statically indeterminate*"; since the stresses in the bars cannot be determined by ordinary statics. For their determination we must have recourse to the elastic properties of the bars or members. Castigliano's Theorem, bringing into play, as it does, the displacement of any joint and the elastic change of form or "deformation" of each bar, may be availed of to prove a second relation or theorem, the direct outgrowth of the first, called the "*Theorem of Least Work*," the application of which to practical cases has the great advantage of involving very simple and direct procedures.

As a preliminary step in the development of this second theorem it will be useful to take up the study and solution of a simple case, from which the general form of the relation is readily inferred; as follows:

27. Statically Indeterminate Frame of Six Bars. In Fig. 9 is shown a frame of six bars, in a vertical plane, forming the sides of a square and *both* diagonals, with one (vertical) load, P_1, as shown, suspended on the lowest pin at e. Evidently there is one redundant bar (bars 1 and 2 are not connected in any manner at A).

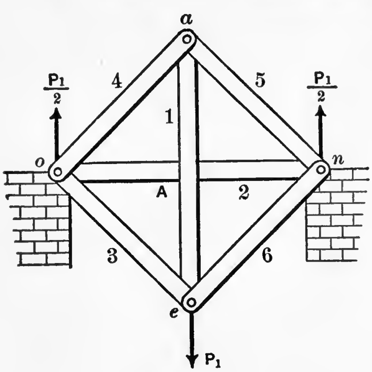

Fig. 9.

The pins at o and n are supported on the horizontal *smooth* surfaces of the *fixed* supports (at the same level), the reactions from which are therefore *vertical* and each equal to $\frac{P_1}{2}$,

since the only load, P_1, is midway between. [Or, pin o might be fixed, and pin n free to slide; the reactions would still be vertical and each would be $\frac{1}{2}P_1$.]

It is assumed, as before stipulated (§ 19), that all of the six bars are of just the proper lengths to enable them to be fitted together without creating stress in any bar. (For example, if all the bars except bar 1 had been put together and bar 1 were too short or too long for its extremities to fit over the pins at a and e, it could not be placed in position without a forcible stretching or shortening, and when this had been accomplished stresses would be created in *all* the bars ("initial stresses"). Such a condition is specially precluded here. The load P_1 being now gradually applied, it is required to determine the final stress T_1 (tension, evidently) in bar 1; and ultimately T_2, T_3, etc.

The method of solution to be used depends on the geometrical fact that the amount of elongation produced in the (vertical) bar 1 must be equal to the difference between the (*downward*) vertical displacements of its extremities, i.e., of joints a and e; since if these displacements were equal it would imply that no change had occurred in the length of bar 1. We shall therefore find an expression for the downward vertical displacement y_a of joint a in terms of the unknown T_1, using the method of §§ 21 and 23 (Castigliano's Theorem); and, similarly, an expression for the downward vertical displacement, y_e, of joint e in terms of T_1; and then subtract the former from the latter and place this difference equal to $\frac{l_1 T_1}{E_1 F_1}$, or $C_1 T_1$, which expresses the elongation λ_1 of bar 1 in terms of T_1. The resulting equation,

$$y_e - y_a = C_1 T_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

will contain but one unknown quantity, viz., T_1, which is then easily determined.

28. Detail of Solution. Conceive bar 1 to be removed, but that its influence on the remainder of the loaded structure is fully and completely maintained by the vertical downward force T_1 (unknown) applied at a and an equal vertical upward force T_1 applied at e. For clearness, however, in

subsequent mathematical work, let us denote the T_1 at a by T, and that at e by T'. We now have Fig. 10 in place of Fig. 9 and note that it shows a structure containing no redundant bars; T and T' playing the part of (unknown) external forces, like P_1 and the two reactions; each of these reactions being now expressed in the form

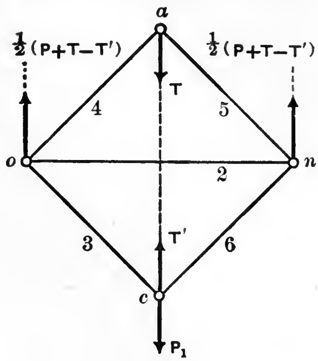

FIG. 10.

$$\tfrac{1}{2}(P_1+T-T').$$

Now the downward vertical displacement of a (y_a, say) is $y_a = \frac{dU}{dT}$; the external force T being vertical and *downward,* so that the desired y_a is the projection, on the line of T, of the actual displacement of joint, or pin, a.

The bars of our present structure being 2 to 6 inclusive, we have [see eq. (29)],

$$y_a = C_2T_2 \cdot \frac{dT_2}{dT} + C_3T_3 \cdot \frac{dT_3}{dT} + \ldots + C_6T_6 \cdot \frac{dT_6}{dT}. \quad . \quad \cancel{(35)} \quad 36$$

Similarly, the displacement of pin e in the *direction* of the external force T' is given by the relation

$$y' = C_2T_2\frac{dT_2}{dT'} + C_3T_3\frac{dT_3}{dT'} + \ldots + C_6T_6\frac{dT_6}{dT'}. \quad . \quad . \quad . \quad (37)$$

But since the force T' points vertically *upward* and y_e denotes the *downward* vertical displacement of pin e, we have

$$y_e = -y'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

From ordinary statics we obtain the values of T_2, T_3, etc., in terms of T and T' and of the load P_1, as shown in the following table (in which the second and third columns, containing certain derivatives to be used later, have been obtained by obvious means):

$T_2=\frac{T}{2}-\frac{P_1-T'}{2}$	$\frac{dT_2}{dT}=\frac{1}{2}$	$\frac{dT_2}{dT'}=\frac{1}{2}$
$T_3=\frac{P_1-T'}{\sqrt{2}}$	$\frac{dT_3}{dT}=0$	$\frac{dT_3}{dT'}=-\frac{1}{\sqrt{2}}$
$T_4=\frac{T}{\sqrt{2}}$	$\frac{dT_4}{dT}=\frac{1}{\sqrt{2}}$	$\frac{dT_4}{dT'}=0$
$T_5=\frac{T}{\sqrt{2}}$	$\frac{dT_5}{dT}=\frac{1}{\sqrt{2}}$	$\frac{dT_5}{dT'}=0$
$T_6=\frac{P_1-T'}{\sqrt{2}}$	$\frac{dT_6}{dT}=0$	$\frac{dT_6}{dT'}=-\frac{1}{\sqrt{2}}$

If now we write T' equal to T and each equal to T_1, and substitute in eqs. (36) and (37) the values just given in the table, we obtain

$$y_a=C_2\left[T_1-\frac{P_1}{2}\right]\cdot\tfrac{1}{2}+0+C_4\frac{T_1}{2}+C_5\frac{T_1}{2}+0; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

and

$$y'=C_2\left[T_1-\frac{P_1}{2}\right]\cdot\tfrac{1}{2}-C_3[P_1-T_1]\cdot\tfrac{1}{2}+0+0-C_6[P_1-T_1]\cdot\tfrac{1}{2}. \qquad (40)$$

Hence the value of y_e-y_a will be, after substituting $-y'$ for y_e [see eq. (38)],

$$-C_2\left[T_1-\frac{P_1}{2}\right]+C_3[P_1-T_1]\cdot\tfrac{1}{2}-C_4\frac{T_1}{2}-C_5\frac{T_1}{2}+C_6[P_1-T_1]\tfrac{1}{2};$$

and this should $=\lambda_1$, i.e., $=C_1T_1$(=elongation of bar 1). That is,

$$C_1T_1+C_2\left[T_1-\frac{P_1}{2}\right]-C_3\cdot\tfrac{1}{2}[P_1-T_1]$$
$$+C_4\frac{T_1}{2}+C_5\frac{T_1}{2}-C_6\left[\frac{P_1-T_1}{2}\right]=0; \quad . \quad . \qquad (41)$$

and finally, solving,

$$T_1=\frac{[C_2+C_3+C_6]P_1}{2C_1+2C_2+C_3+C_4+C_5+C_6}. \quad . \quad . \quad . \qquad (42)$$

(For instance, if the coefficients C_1, C_2, C_3, etc., were *all equal* we should obtain $T_1 = \frac{3}{8} P_1$.)

But we shall now show that eq. (41) is nothing more than what may be obtained by writing $\frac{dU}{dT_1} = 0$; where U is the *total "internal work" of the whole* framework, that is, of all the six bars, *including bar* 1 *itself.*

Since $U = \frac{1}{2}C_1T_1^2 + \frac{1}{2}C_2T_2^2 + \ldots + \frac{1}{2}C_6T_6^2$, what we understand by $\frac{dU}{dT_1}$ is

$$\left[C_1T_1\frac{dT_1}{dT_1} + C_2T_2\frac{dT_2}{dT_1} + \ldots + C_6T_6\frac{dT_6}{dT_1}\right]; \quad . \quad . \quad (43)$$

Hence, utilizing the table following eq. (38), where we find T_2, T_3, etc., as functions of T and T', each of which will now be replaced by its real value, T_1, so that $\frac{dT_2}{dT_1} = \frac{dT_2}{dT} + \frac{dT_2}{dT'}$, etc., we find

$$\left.\begin{array}{l} \frac{dU}{dT_1} = C_1T_1 + C_2\left[T_1 - \frac{P_1}{2}\right](\frac{1}{2}+\frac{1}{2}) + C_3\frac{P_1 - T_1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) \\ \quad + C_4\frac{T_1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + C_5 \cdot \frac{T_1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + C_6\frac{P_1 - T_1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) \end{array}\right\}, \quad . \quad (44)$$

and if this expression be written equal to zero, we obtain

$$C_1T_1 + C_2\left[T_1 - \frac{P_1}{2}\right] - C_3\frac{P_1 - T_1}{2} + C_4\frac{T_1}{2} + C_5\frac{T_1}{2} - C_6 \cdot \frac{P_1 - T_1}{2} = 0; \quad (45)$$

which is identical with eq. (41) and will yield the correct value of T_1.

But the value of T_1 obtained from eq. (45) is a value which, from the principles of the differential calculus, would cause U, the total internal work of the structure, to be a *minimum;* that is, of all values that could be proposed for T_1, that one is the true and only possible one which makes the expression for U a minimum.* This is called the **principle of least work,**

* Sinc. $dU/aT_1 = 0$. See Note C, Appendix.

first established by Castigliano in 1878; and it will now be proved in its general form.

29. Principle of Least Work. General Proof. (For a loaded elastic structure containing one redundant bar; all bars closely fitted originally *without initial stress.*) Select the bar to be considered as the redundant bar and let T_1 denote the final stress in it. In Fig. 11 *ae* represents this bar before the structure is loaded. As a result of the loading the joint a has moved to a'' and the other extremity of the bar, viz., joint e, has moved to e''. Perpendiculars having been let fall from a'' and e'' upon the original position, *ae*, of the bar (when unstrained), we note that $y_a, = aa'$, is the displacement of joint a in the direction of the bar, and that $y_e, = ee'$, is the displacement of joint e in the direction of the bar. If we suppose the bar to have stretched as a result of the application of the loading evidently the difference

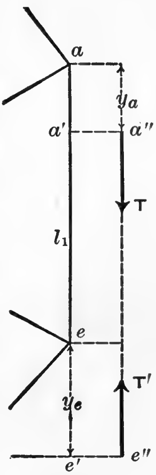

FIG. 11.

$y_e - y_a$ is the elongation, $= \lambda_1$, $= \dfrac{l_1 T_1}{F_1 E_1}$, $= C_1 T_1$, i.e.,

$$y_e - y_a = C_1 T_1, \quad . \quad . \quad . \quad . \quad (46)$$

where T_1 is the final stress in the bar.

Now conceive the bar to be removed, the forces which it exerts on the joints a and e at its extremities being applied at those joints, respectively, to preserve the remaining bars in the same position and state of stress as before. These forces are equal, and act in the same line, but point in opposite directions. For this case (bar in tension) they point toward each other (see Fig. 11) and will be called T and T' respectively, although they are actually equal, each being equal to T_1. For the remaining bars T and T' play the part of *external forces;* i.e., either T or T' might be conceived to vary, i.e., change its value, the other remaining constant, without destroying the equilibrium of the structure formed by the remaining bars (note that such a statement as this *could not be made if the bar in question were not a redundant bar*).

The remaining bars constituting a statically determinate

system, then, with T and T' playing the part of independent external forces, and the internal work of this system being denoted by U'', the result of performing the operation $\frac{dU''}{dT}$ would be an expression for the displacement of joint a in the direction of T, i.e., for y_a, and no change of sign is necessary, since joint a has been displaced in the direction in which T points; but the expression given by the operation $\frac{dU''}{dT'}$ would have to be taken with a *contrary sign* to give the displacement y_e, since joint e has been displaced in a direction (measured parallel to T') *opposite* to that in which T' points. That is, otherwise expressed, we have

$$y_a = \frac{dU''}{dT}; \quad \text{and} \quad y_e = -\frac{dU''}{dT'}. \quad . \quad . \quad . \quad . \quad (47)$$

Hence eq. (46) becomes

$$-\frac{dU''}{dT'} - \frac{dU''}{dT} = C_1T_1; \quad \text{i.e.,} \quad \frac{dU''}{dT} + \frac{dU''}{dT'} + C_1T_1 = 0. \quad (48)$$

Now since $T = T' = T_1$, the terms $\frac{dU''}{dT} + \frac{dU''}{dT'}$ are the same thing as $\frac{dU''}{dT_1}$, i.e., the complete derivative of U'' with respect to T_1, U'' being the internal work of all the bars except bar 1; while C_1T_1 is nothing more than $\frac{d}{dT_1}\left[\frac{C_1T_1^2}{2}\right]$, i.e, the derivative of the internal work of bar 1 itself, with respect to T_1. Hence the left-hand member of eq. (48) is simply the derivative, with respect to stress T_1 in bar 1, of the *internal work of all the bars*, including bar 1, i.e., of the total internal work, U, of the original complete system of bars. Eq. (48) then, takes the form

$$\frac{\mathbf{dU}}{\mathbf{dT_1}} = \mathbf{0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

as the condition or equation the solution of which will give the value of the stress T_1 in bar 1.

In other words, of all values that could be proposed for T_1 the actual or true value is the one which would make the function U, or *total internal work*, a *minimum*. Hence the term, **Principle, or "Method, of Least Work."** (Footnote, p. 30.)

N.B. If the extremities of bar 1, supposed to be in tension in above proof, were displaced in directions contrary to those assumed; or if the bar were in compression, no matter what the actual movement of the extremities, in space, due to the application of the loading; it is easily shown that the same result would be reached [eq. (49)]. This result is therefore general.

29a. Theorem of "Least Work" a Particular Case of a More General Relation. The use of eq. (49) has already been illustrated in solving the statically indeterminate frame of § 28. It should be noted, however, that the idea of "Least Work" which has given rise to the name of the method is purely an incidental matter, of scientific, rather than practical, interest; since the proof of eq. (49) does not depend upon this idea, but is based on the ordinary relations of geometry and elasticity in connection with Castigliano's Theorem for displacements of the joints of a statically determinate structure. This same mode of proof will be employed later (§ 32) in establishing a method of dealing with a structure containing redundant bars which are initially too long, or too short, to fit the joints already determined by the "necessary" bars of the structure; and it will be seen that eq. (49) is merely a particular case of this more general relation.

30. Statically Indeterminate Structure Containing *Two* Redundant Bars. The truss in Fig. 12, resting on two smooth horizontal piers at the same level, consists of one square panel and one rectangular panel, involving eleven bars fitted originally (i.e., before the loading) *without strain*. There are three equal loads, each equal to P, applied at the three upper joints. Evidently there are two redundant or "unnecessary" bars; that is, no more than two bars could be omitted without collapse. For example, bars 5 and 9 could be omitted; or bars 5 and 2; and the remainder of the structure would hold its shape and form a "statically determinate" structure. (In this particular case it also happens that, since bars 6 and 11 are in the same

line and there is no load at their junction, the three bars 1, 2, and 7 might be omitted without collapse.)

Let us imagine bars 1 and 2 removed and the (unknown) stresses in them (suppose each to be a tension), T_1 and T_2, to act at the proper joints on the remainder of the truss, this

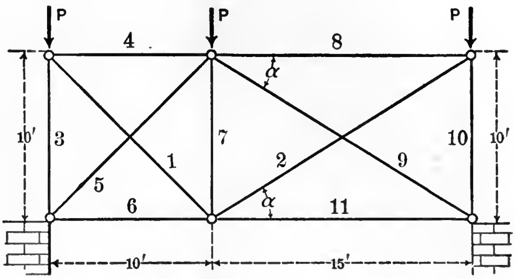

FIG. 12.

remainder being now a "*statically determinate*" structure under the nine external forces (see Fig. 13), T_1 in two places, T_2 in two places, P in three places (loads), and the two vertical reactions $\frac{8}{5}P$ and $\frac{7}{5}P$.

We can now by simple statics determine the values of the internal stresses T_3, T_4, etc., up to T_{11} (inclusive) in terms of these "external forces" just mentioned. These values are assembled in the following table and also (for a future purpose)

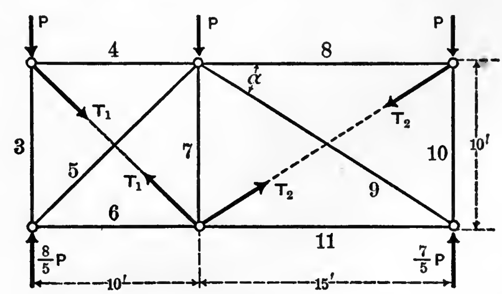

FIG. 13.

values of the derivatives of each of these internal stresses with respect to T_1 and to T_2. We here denote $\cos \alpha$ by m, and $\sin \alpha$ by n (α itself is $33° 40'$).

$$\left.\begin{array}{l|l|l}
T_3=P+\dfrac{T_1}{\sqrt{2}} & \dfrac{dT_3}{dT_1}=\dfrac{1}{\sqrt{2}} & \dfrac{dT_3}{dT_2}=0 \\[2ex]
T_4=\dfrac{T_1}{\sqrt{2}} & \dfrac{dT_4}{dT_1}=\dfrac{1}{\sqrt{2}} & \dfrac{dT_4}{dT_2}=0 \\[2ex]
T_5=T_1-\dfrac{3}{5}P\sqrt{2} & \dfrac{dT_5}{dT_1}=1 & \dfrac{dT_5}{dT_2}=0 \\[2ex]
T_6=\dfrac{3}{5}P-\dfrac{T_1}{\sqrt{2}} & \dfrac{T_6}{dT_1}=-\dfrac{1}{\sqrt{2}} & \dfrac{dT_6}{dT_2}=0 \\[2ex]
T_7=\dfrac{T_1}{\sqrt{2}}+nT_2 & \dfrac{dT_7}{dT_1}=\dfrac{1}{\sqrt{2}} & \dfrac{dT_7}{dT_2}=n \\[2ex]
T_8=mT_2 & \dfrac{dT_8}{dT_1}=0 & \dfrac{dT_8}{dT_2}=m \\[2ex]
T_9=T_2-\dfrac{2}{5}\dfrac{P}{n} & \dfrac{dT_9}{dT_1}=0 & \dfrac{dT_9}{dT_2}=1 \\[2ex]
T_{10}=P+nT_2 & \dfrac{dT_{10}}{dT_1}=0 & \dfrac{dT_{10}}{dT_2}=n \\[2ex]
T_{11}=\dfrac{2}{5}\cdot\dfrac{m}{n}P-mT_2 & \dfrac{dT_{11}}{dT_1}=0 & \dfrac{dT_{11}}{dT_2}=-m
\end{array}\right\} . \qquad (50)$$

Let us now restore bar 2 to the truss in Fig. 11. The result is a new truss with *one* redundant bar (taken as bar 2) the stress in which (T_2) can therefore be found in terms of T_1 and P by § 29; since for this new truss the external forces are the two T_1's, the three P's, the $\frac{8}{5}P$ and the $\frac{7}{5}P$; (T_1 is yet unknown, of course).

We now put $\dfrac{dU_{2\,\ldots\,11}}{dT_2}=0$, with the understanding that $U_{2\,\ldots\,11}$, denotes the total internal work of this new truss, consisting of bars 2 to 11 inclusive; that is,

$$U_{2\,\ldots\,11}=\frac{C_2T_2^2}{2}+\frac{C_3T_3^2}{2}+\ldots+\frac{C_{11}T_{11}^2}{2}, \quad . \quad . \quad (51)$$

whence

$$\frac{dU_{2\,\ldots\,11}}{dT_2}=C_2T_2\frac{dT_2}{dT_2}+C_3T_3\frac{dT_3}{dT_2}+\ldots+C_{11}T_{11}\frac{dT_{11}}{dT_2}, \;=0. \qquad (52)$$

Filling out details of eq. (52) by the aid of the table, or eqs. (50), there is obtained, after reduction,

$$\frac{nC_7T_1}{\sqrt{2}}+[C_2+n^2C_7+m^2C_8+C_9+n^2C_{10}+m^2C_{11}]T_2$$
$$-[\tfrac{2}{5}C_9-n^2C_{10}+\tfrac{3}{5}m^2C_{11}]\frac{P}{n}=0, \quad \ldots \ldots \quad (53)$$

which contains two unknowns, viz., T_1 and T_2.

Similarly, restore bar 1 to the truss in Fig. 13, and we again have a new truss with only one redundant bar (taken as bar 1). The new truss consists of bar 1 and bars 3 to 11 inclusive, the forces external to it being the two T_2's, the three P's and the $\frac{8}{5}P$ and $\frac{7}{5}P$. Its total internal work, which may be denoted by $U_{1,\,3\,\ldots\,11}$,

$$=\frac{C_1T_1^2}{2}+\frac{C_3T_3^2}{2}+\frac{C_4T_4^2}{2}+\ldots+\frac{C_{11}T_{11}^2}{2}; \quad . \quad . \quad (54)$$

and by putting its first derivative with respect to T_1 equal to zero we obtain [by eq. (49)] an expression for T_1 in terms of the external forces just mentioned. Now

$$\frac{dU_{1,\,3\,\ldots\,11}}{dT_1}=C_1T_1\frac{dT_1}{dT_1}+C_3T_3\frac{dT_3}{dT_1}$$
$$+C_4T_4\frac{dT_4}{dT_1}+\ldots+C_{11}T_{11}\frac{dT_{11}}{dT_1}; \quad . \quad (55)$$

and this is to be placed equal to zero. Therefore, substituting details from eqs. (50), we have, after simplification,

$$\left[C_1+\frac{C_3+C_4}{2}+C_5+\frac{C_6+C_7}{2}\right]T_1+\frac{nC_7}{\sqrt{2}}T_2$$
$$+[C_3-\tfrac{8}{5}C_5-\tfrac{3}{5}C_6]\frac{P}{\sqrt{2}}=0. \quad . \quad (56)$$

We now have two equations, viz., (53) and (56), from which by ordinary algebra both T_1 and T_2 can be determined.

If we take, as a particular instance of this example, the value of 45° for α, we must put $\sqrt{2}\div 2$ both for m and n, while $\frac{1}{2}P$ replaces the $\frac{3}{5}P$ and also the $\frac{2}{5}P$ in eqs. (50), T_1 will also be equal to T_2, from symmetry; and $\therefore$ eq. (56) alone will

serve to determine T_1, which in this way is found to be (with all the C's taken equal)

$$T_1, = T_2, = \frac{P}{9\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

31. Statically Indeterminate Structure with More than Two Redundant Bars. (The usual stipulation is made, that all the bars are closely fitted originally, without strain.) With three redundant bars, say, bars 1, 2, and 3, we have simply to replace them by the pairs of (unknown) forces which they exert on the joints at their extremities, and by applying ordinary statics obtain the stresses T_4, T_5, etc., in all the other bars in terms of the unknown T_1, T_2, and T_3, and the external forces of the original truss. The derivatives of each of the stresses T_4, T_5, etc., are then obtained with respect to T_1, to T_2, and to T_3, and set down in a table.

Bar 1 is then replaced in the truss, which thus becomes a truss of one redundant member (viz., bar 1) its external forces consisting of the two T_2's, the two T_3's, and the external forces of the original truss. By the use of eq. (49) an expression is obtained for T_1 in terms of T_2, T_3, and the original external forces.

By replacing bar 2 in the truss instead of bar 1 and using eq. (49), a second equation is obtained giving T_2 in terms of T_1 and T_3, and the original external forces. Similar treatment with bar 3 gives a third equation furnishing T_3 in terms of T_1, T_2, and the original external forces. From these three independent equations the values of the three quantities T_1, T_2, and T_3 are then easily determined.

The procedure is now evident for the case of any number of redundant bars, but manifestly involves a large amount of detail when there are more than two such bars.

32. Statically Indeterminate Structure with One Redundant Bar, this Bar being Originally Too Short, or Too Long, to Fit into its Place. In a structure of one redundant bar let us suppose that the "necessary" bars have first been fitted together and that is it then found that the remaining (redundant) bar is λ_0 inches longer, or shorter, than the distance between the joints which it is to connect; so that to adjust it to its

place it is necessary by external means to force the two joints nearer to, or further from, each other, and either to stretch or shorten the bar itself, to cause its extremities to fit over the pins at these joints. This being done, and the external constraint removed, the structure is provided with some character of support, and is loaded at one or more joints. The small distance λ_0 (surplus or deficiency, in length; of the bar in question) is supposed to be small; so that when the bar is forced into position and the loads placed, the elastic limit is not passed in any bar. We are now to prove, denoting by U the total internal work of the structure (*all* the bars) in its final condition under load, and by T_1 the stress in the redundant bar (bar 1), that

$$\frac{dU}{dT_1} = \lambda_0, \qquad (57)$$

for the case where bar 1 is *assumed to be in tension* and is originally *too short.*

A specific case will be taken. In Fig. 14 the five "necessary" bars (2, 3, 4, 5, and 6) when fitted together form the

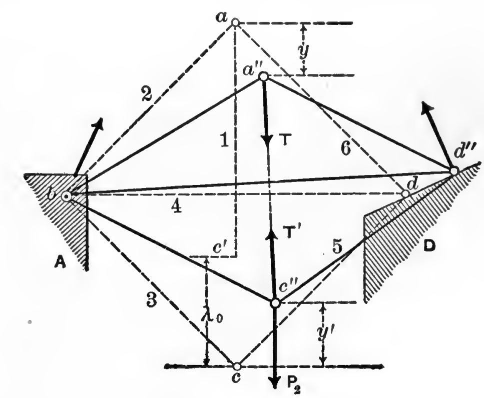

FIG. 14.

quadrilateral figure (dotted lines) *abcd*; bar 4 being a diagonal, *bd*. Bar 1, whose unstrained length is *ac'*, is to form the other diagonal; that is, it is to connect the joints a and c; but is *too short*, by an amount $\overline{c'c}$, or λ_0 inches. Joint b is fixed on the support A, while the pin of joint d is free to slide along the

smooth hard surface of the fixed support D. Let now the lower extremity of bar 1 (ac') be forcibly fitted over the pin of joint c and the load P_2 applied at the latter joint. This special external constraint having been removed, and the load P_2 being in place, we find joint d in a new position, d'', on the supporting surface of D; while joints a and c are now found, respectively, at a'' and c''. Bar 1 is in a state of tension, the stress in it being T_1, unknown. Its unstrained length being $\overline{ac'}=l_1$, its present length is $\overline{a''c''}=l+\lambda_1$, where λ_1 is its elongation.

Let us now consider bar 1 to be removed, its action on the two joints a'' and c'' being fully represented by the forces T at a and T' at c'', these forces being in the same line $a''c''$, each equal to the unknown T_1, and pointing *toward each other*, as shown. They may now be considered as external forces (and independent of each other), acting on the remaining bars, viz., 2, 3, 4, 5, and 6; these bars forming a statically determinate structure which has been changed from the form $abcd$ (dotted lines) into the form $a''bc''d''$ (full lines) by the action of the external forces T, T', and P_2; the supports being of the nature described.

Hence, if U'' denote the internal work for all the bars *except bar* 1, we have, from Castigliano's Theorem (§ 21) the displacement of a in the direction of force T, viz., the projection marked y in the figure is

$$y=+\frac{dU''}{dT}, \quad \dots \dots \dots \quad (58)$$

and similarly, the displacement of c in direction of force T', or the projection y' of the figure, is

$$y'=+\frac{dU''}{dT'}. \quad \dots \dots \dots \quad (59)$$

(Note that for this figure both of the above signs are $+$, since each of the displacements concerned takes place in the direction in which the corresponding force points.)

But, from the geometry of the figure, $\overline{ac'}+\lambda_0=y+\overline{a''c''}+y'$; i.e.,

$$l_1+\lambda_0=y+l_1+\lambda_1+y'. \quad \dots \dots \quad (60)$$

Introaucing into (60) the relation $\lambda_1 = \frac{l_1 T_1}{F_1 E_1}, = C_1 T_1$, and the values of y and y' from (58) and (59), we have

$$\frac{dU''}{dT} + \frac{dU''}{dT'} + C_1 T_1 = \lambda_0. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Now T and T' each $= T_1$ and $C_1 T_1$ is nothing more than the derivative of the internal work of bar 1, viz., $\frac{1}{2} C_1 T_1{}^2$, with respect to T_1. Hence the left-hand member of eq. (61) is the complete derivative of the internal work, U, of *all the bars* (including bar 1) and (61) may therefore be written

$$\frac{dU}{dT_1} = \lambda_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The value of the stress T_1 may therefore be determined by filling out eq. (62) and solving for T_1; i.e., by first expressing the stress in each bar in terms of T_1 (use being made of the "free body" (full lines) in Fig. 14; the values T and T' being now replaced by T_1); and then writing out the separate derivatives needed (besides the stresses thesmelves) for substitution in the six terms in the left-hand member of eq. (62), i.e., in the expression

$$C_1 T_1 + C_2 T_2 \frac{dT_2}{dT_1} + C_3 T_3 \frac{dT_3}{dT_1} + \ldots + C_6 T_6 \frac{dT_6}{dT_1} = \lambda_0. \quad . \quad (63)$$

While it is evident that in case bar 1 is originally too short to fit into its place it will be found in a state of tension after it has been forcibly fitted into place and the truss left to itself without load; nevertheless, its place in the truss and the position of the load or loads may be such that afterwards, when the truss is loaded, it will be found in a state of *compression*, instead of tension. This result would be evidenced by the obtaining of a negative result for the stress T_1 in the *final solution* of eq. (63); that is, T_1 is thus found to be a "negative tension," or compressive stress.

Bar 1 Originally too Long. If the redundant bar (bar 1) is originally *too long*, by λ_0 inches, to connect joints a and c of Fig. 14, point c' will be *below* c''. We have, therefore, with bar 1 in its final position and the truss loaded, the bar being

assumed in a state of compression, and its final shortening being $\lambda_1 = C_1T_1$,

$$\overline{ac}, = l - \lambda_0, = y + (l - \lambda_1) + y'. \quad . \quad . \quad . \quad . \quad (64)$$

If we now suppose bar 1 to be removed, the forces T and T' (at a'' and c'') taking its place for equilibrium, these two forces must be inserted pointing *away from*, instead of toward, each other; so that we now have

$$y = -\frac{dU''}{dT} \quad \text{and} \quad y' = -\frac{dU''}{dT'}. \quad . \quad . \quad . \quad . \quad (65)$$

Hence, by combining, we reach the same result as in the case of the bar being too short, viz.,

$$\frac{dU''}{dT} + \frac{dU''}{dT'} + C_1T_1 = \lambda_0; \quad . \quad . \quad . \quad . \quad . \quad (66)$$

or

$$\frac{dU}{dT_1} = \lambda_0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

but it must be noted, that in the use of this relation for the case of bar 1 *too long*, the stress in the bar *must be assumed to be compressive* and the determination of stresses in the other bars in terms of T_1 must be made *on this basis*. (If the actual stress in bar 1 is tensile the fact will be brought out by the obtaining of a negative value for T_1 in the final numerical result.)

It will be noted that when $\lambda_0 =$ zero for the redundant bar eqs. (62), (63), and (67) reduce to what has been called the *Theorem of "Least Work"* (see § 29*a*).

33. Example of Single Redundant Bar, when Originally Too Short. For this we may take the frame of six bars already shown in Fig. 9 (p. 26), § 27, and suppose that originally bar 1 was *too short*, by λ_0 inches, to connect joints a and e. In the treatment of the problem in § 27 the stresses in the other bars were found in terms of T_1 (there called T and T', temporarily) on the assumption that the final state of bar 1 is *tension*; which is suitable for present purposes in the use of eq. (57), or (63), according to which we have only to set the

expression in eq. (44) equal to λ_0, instead of zero. This being done, the result is

$$T_1 = \frac{\lambda_0 + [C_2 + C_3 + C_6]P_1}{2C_1 + 2C_2 + C_3 + C_4 + C_5 + C_6}, \quad . \quad . \quad . \quad (68)$$

which may now be compared with the value in eq. (42) for the case where bar 1 could be fitted in place originally without strain.

34. Statically Indeterminate Structure with Two or More Redundant Bars; Each of which is Too Long, or Too Short, to Fit into Place. A case of two such bars will suffice to indicate the method for any other number. Let the structure contain nine bars, of which only seven are "necessary" bars, the other two being redundant. Let the bars to be taken as the the redundant bars be 4 and 5; bar 4 being originally $(\lambda_0)_4$ inches too short, and bar 5 $(\lambda_0)_5$ inches too long, to fit into place. Assume, therefore, that the final state of 4 is tension; and that of 5, compression.

Let now the redundant bars 4 and 5 be removed from the truss, and the forces which they exerted on the joints at their extremities put in; T_4 at two joints, and T_5 at two joints, pointing *in proper directions* to correspond with the assumption of *tension* or *compression* in the corresponding bars. We now have a statically determinate structure consisting of the seven bars 1, 2, 3, 6, 7, 8, and 9; the stresses in which are now obtained in terms of the loads and of the unknown T_4 and T_5, by ordinary statics. Now conceive bar 4 to be again in place (but not bar 5), and we have a statically indeterminate structure, with *only one* redundant bar (bar 4), the external forces for which are the two T_5's, the loads, and the reactions of supports; in terms of which forces, T_4 may now be obtained by the use of eq. (62), viz.:

$$\frac{d[U_{1,\,2,\,3,\,4;\;6,\,7,\,8,\,9}]}{dT_4} = (\lambda_0)_4, \quad . \quad . \quad . \quad . \quad (68)$$

where the left member consists of terms of the type $CT\frac{dT}{dT_4}$, one for each of the original bars, *except* 5.

Again, let bar 5 be put back in the truss, and bar 4 taken out; and the structure thus obtained contains only one redundant bar (which now is bar 5) and is acted on by external forces, consisting of the two T_4's, the loads, and the reactions of supports. Hence from eq. (67),

$$\frac{d[U_{1,\ 2,\ 3;\ 5,\ 6,\ 7,\ 8,\ 9}]}{dT_5}=(\lambda_0)_5, \quad . \quad . \quad . \quad . \quad (69)$$

where the left-hand member consists of terms of the type $CT\frac{dT}{dT_5}$, for one each of the bars of the original truss, *except* 4. Eq. (69) will give the stress T_5 in terms of the external forces just mentioned.

Equations (68) and (69), then, are two simultaneous equations containing the two unknown quantities T_4 and T_5; and the values of the latter are now obtained by ordinary elimination. Should a negative number be obtained as the value of either T_4 or T_5 it indicates that the character of the stress in the bar is the opposite of that originally assumed.

The procedure to be adopted in case there are more than two redundant bars is now evident. There will be as many independent equations like (68) or (69) as there are redundant bars. As a basis for each one of these equations we have under consideration a truss with only one redundant bar, the forces external to this truss being the original loads, the reactions of supports, and the forces exerted (on the joints at their extremities) by all the *other* (original) redundant bars. Attention is again called to the necessity of assuming each of the original redundant bars to be in *tension*, or in *compression*, according to whether it is *too short*, or *too long* (respectively) to fit into place after the "necessary" bars have been put together.

35. Temperature Stresses in a System of Elastic Bars. The foregoing treatment finds application in the following circumstances: A *change* of temperature from that common to all the bars at the time of assembling the truss, if we suppose an original close fit, without strain, at that temperature, may create stresses in the bars *without any load being placed on the truss.*

There may be redundant bars, or the character of the supports may be such, that when the temperature rises (or falls), the lengths natural to the bars at the new temperature (with no load on truss) are not those that they are constrained to have in their connection with each other; so that if the truss were reassembled at the new temperature the bars would not fit into place without special external (temporary) constraint; that is, after the "necessary" bars had been put together at the new temperature each of the others would be found to be too short, or too long, to fit into place without such external constraint.

Hence we may determine the stresses in these latter (redundant) bars (and ultimately in all the bars), after noting by how much each such bar is too short, or too long, to fit into its place in the assemblage of the "necessary" bars, by applying the methods of §§ 32, 33, and 34. Stresses thus induced are called *temperature stresses.* These stresses are usually obtained independently of the action of loads, and combined later with those due to the loading.

CHAPTER IV

INTERNAL WORK IN BEAMS UNDER FLEXURE. APPLICATIONS

36. Internal Work of a Bent Beam. Straight Beams; and Curved Beams of Large Radius of Curvature. When a beam originally straight, or curved (and in that case the radial thickness must be small compared with the radius of curvature), is slightly deformed by being made the means of holding in equilibrium a system of applied forces and reactions (all acting in the same plane, containing the axis of the beam), each slice or lamina (of the beam) bounded by two imaginary cutting planes at right angles to the axis of the beam and a certain small distance apart denoted by ds, measured along that axis, consists of a bundle of fibers (so called) on the extremities of which (in the two cutting planes), by the common theory of flexure, we find elastic forces acting, viz., tensile, or compressive, and shearing. In general [see Fig. 15, (A)], we find at each end of the lamina, or block, a total shear represented by J (lbs.) (or J'), a total thrust, T (lbs.) (or T'), consisting of a set of uniformly distributed compressive or tensile forces of uniform intensity p_1 (lbs. per sq.in.); and a set of tensions and compressions forming a " stress-couple " whose moment (" bending moment ") is denoted by M and is equal to $\frac{p_2 I}{e}$ (in which p_2 denotes the stress, lbs. per sq.in., in the outer fiber at a distance e from the center of gravity, or axis, of the cross-section, and I the " *moment of inertia* " of the plane figure (formed by the cross-section) with reference to the gravity axis G, perpendicular to the paper. dF denotes an elementary area of the cross-section and z is its distance from the gravity axis G. (See p. 354 of the author's " Mechanics.")

Before stress, the block had the unstrained shape NHH_0N_0. Under stress, however, its shape is $N''H''H_0N_0$, which new shape we may consider to have come about thus: The uniformly distributed thrust T causes all the fibers to shorten equally by

an amount $d\lambda_1$, thus bringing the face NH to a parallel position $N'H'$; then the couple acts, causing the fibers to shorten or lengthen unequally, but proportionally to the distance z from the gravity axis G of the cross-section (in which operation the outer fiber NN_0 shortens by an amount $d\lambda_2$); then the shearing stress comes into play to distort slightly the corner angles of the small blocks into which each fiber may be conceived to be

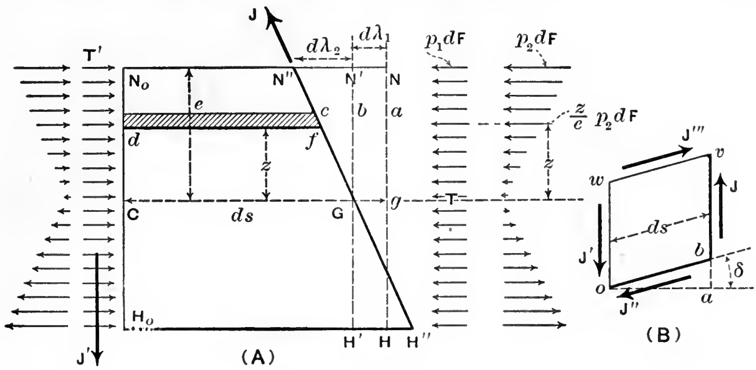

FIG. 15.

subdivided (no attempt has been made to show this distortion in the figure).

We have therefore,

The work of thrust on ad, or one fiber, $=\frac{1}{2}p_1dF\,.\,\overline{ab}=[\frac{1}{2}p_1dF]d\lambda_1$;

but, from definition, $\qquad \dfrac{d\lambda_1}{ds}=\dfrac{p_1}{E}$;

hence the work of thrust (one fiber), $=\frac{1}{2}p_1dF\dfrac{p_1ds}{E}=(\frac{1}{2}p_1{}^2dF)\dfrac{ds}{E}$;

hence the whole work of thrust for all the fibers composing the lamina is (with F denoting whole area of section $N''H''$),

$$\tfrac{1}{2}p_1{}^2ds\frac{F}{E}=\tfrac{1}{2}\frac{(Fp_1)^2ds}{FE}=\tfrac{1}{2}\,.\,\frac{T^2}{FE}ds. \quad . \quad . \quad . \quad (70)$$

As to the work of the stress-couple, each fiber shortens an amount $\overline{bc}$, $=\dfrac{z}{e}$ of $d\lambda_2$, due to a force $\dfrac{z}{e}p_2dF$; and hence the work on one fiber $=\left(\frac{1}{2}\,.\,\dfrac{z}{e}p_2dF\right)\left(\dfrac{z}{e}d\lambda_2\right)$. But $\dfrac{d\lambda_2}{ds}=\dfrac{p_2}{E}$, and

therefore the work on one fiber, as due to the stress-couple, is $\frac{1}{2} \cdot \frac{p_2{}^2 ds}{e^2} \cdot \frac{dFz^2}{E}$; and therefore the

$$\left.\begin{array}{l}\text{Whole work due to stress-couple}\\ \quad\text{for the whole lamina or block}\end{array}\right\} = \frac{1}{2} \cdot \frac{p_2{}^2 ds}{Ee^2} \int_{H''}^{N''} dFz^2 = \frac{ds}{2} \cdot \frac{p_2{}^2}{Ee^2} I;$$

which can be written $\frac{ds}{2EI}\left(\frac{p_2 I}{e}\right)^2$. But $\frac{p_2 I}{e}$ is M, *the moment of the stress-couple* (or the "*bending moment*") and hence the last expression becomes

$$\frac{1}{2} \cdot \frac{M^2 ds}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

As for the work of the shear, J, if the whole block were gradually distorted from a square-cornered condition to a rhombus [δ being the change of angle at the corner. See Fig. 15 (B)] by the shearing couple whose moment is Jds we should have the work of J (all the other forces being neutral) $= \frac{1}{2} \cdot J\overline{ab} = \frac{J}{2}(ds)\delta$, where δ is the angle of shearing distortion.

Now (by definition) $\delta = \frac{J}{F} \div E_s$, where E_s is the modulus of elasticity for shearing (p. 227, M. of E.), and therefore the whole work due to shear for this lamina or block would be $\frac{1}{2} \cdot \frac{J^2}{FE_s} \cdot ds$. But since the shear is not uniformly distributed over the cross-section in any case of flexure, this last expression must be modified by the use of a coefficient, or multiplier, A, to suit different forms of cross-section. For example, for a solid rectangular cross-section with two bases parallel to the "force plane" or plane of the paper, it can be proved that $A = \frac{6}{5}$; while for a solid circular section $A = \frac{10}{9}$. For an *ordinary* I-*beam* with a thin web, if the web be considered as bearing most of the shear, A becomes practically *unity* (but then F is the area of the section of the *web alone*). Hence, the work of the shear for one lamina, of length ds, would be

$$\frac{A}{2} \cdot \frac{J^2 ds}{FE_s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

The forces shown in Fig. 15 (A), as acting on the slice or lamina situated between two neighboring cross-sections, play the part of external forces for this elastic body. But since this elastic body may be considered to be made up of an immense number of elastic bars pivoted to each other, these bars being gradually stretched or shortened during the gradual application of external forces, the total internal work performed in this operation will be equal in value to the sum of the three terms just found, viz.:

$$dU=\tfrac{1}{2}\cdot\frac{T^2ds}{FE}+\tfrac{1}{2}\frac{M^2ds}{EI}+\frac{A}{2}\frac{J^2ds}{FE_s}. \quad . \quad . \quad . \quad (73)$$

Now an entire beam, straight or curved, may be considered to be made up of a great number of consecutive slices or laminæ like that in Fig. 15 (A), each having a length ds along the axis of the beam and subjected to a thrust T at each end; to two stress-couples, one at each end, of a moment M; and to two shears, one at each end. Consequently, the total internal work of such a beam can be expressed as

$$U=\left[\int\frac{T^2ds}{2FE}+\int\frac{M^2ds}{2EI}+\int\frac{AJ^2ds}{2FE_s}\right] \quad . \quad . \quad . \quad (74)$$

(the integration being extended along the whole length of the beam) and may be placed equal to the sum of the amounts of external work of the external forces applied to the beam, or "applied forces" (including reactions) to which the state of stress of the beam is due.

37. Displacement of a Point in Axis of a Straight Prismatic Beam in Flexure. The expression for U in eq. (74), having been once found in any given case, may be used in Castigliano's Theorem [§21, eq. (19)] for determining deflections or displacements of any point of the axis of a beam where there is a load applied (or where a load may be imagined to be applied and then reduced to zero, as indicated in § 23a). The term due to shear is generally of very small consequence and may therefore be omitted in many cases without sensible error. The term involving the thrust is also sometimes negligible, according to circumstances. The beams concerned being usually of homogeneous material, the quantity E may then be taken outside

the integral sign; and, furthermore, if all cross-sections are alike and similarly placed, the "moment of inertia," I, of the cross-section will be the same at all sections and consequently may in such cases be placed outside the sign of integration. Frequently when it is wished to differentiate this expression for U with reference to some quantity which it will eventually contain (say, one of the external forces P), the operation may be shortened by differentiating the general expression under the integral sign; thus (in the term involving M)

$$\frac{d}{dP}\int_0^l \frac{M^2 ds}{2EI} = \int_0^l \frac{2M}{2}\left(\frac{dM}{dP}\right)\frac{ds}{EI} = \int_0^l M\left(\frac{dM}{dP}\right)\frac{ds}{EI}. \quad (75)$$

Hence, when (later) M is expressed as a function of P it will simply be necessary to find M, and $\left(\frac{dM}{dP}\right)$, in terms of P and substitute in eq. (75); thus dispensing with the squaring of the M as called for literally in eq. (74). A similar remark applies to the other general terms, involving the thrust and the shear, respectively.

38. Example I. Horizontal, Straight, Prismatic Beam. End Supports. Load in Middle. Central Deflection Required. The (originally) straight, prismatic, homogeneous beam is placed on two supports at the same level and loaded in the middle with a concentrated load P, vertical. Evidently the reaction at each pier is $V = P \div 2$ (Fig. 16) and is vertical

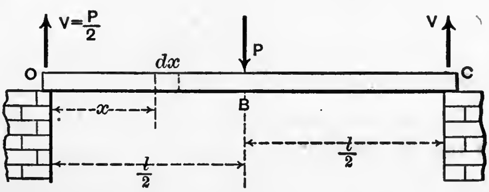

FIG. 16.

(smooth horizontal surfaces); and the thrust is *zero* at all sections; while the work of the shear will be neglected, as being very small.

It is required to find the vertical deflection y_1 for the middle point B of the axis of the beam, as due to the gradual application of the one load P, applied at B. Neglect the weight of

the beam itself. Evidently the deflection desired is the displacement of the point of application of P in the direction of P itself and therefore can be obtained by an application of eq. (25) (i.e., Castigliano's Theorem).

In this case we shall measure the abscissa, x, of any dx in the left-hand half of the beam, from the extremity O; and since the beam is straight, dx will take the place of ds in the general eq. (74). The "free body" of length x is acted on by the reaction $\frac{1}{2}P$ at one end and by the shear J and the tensions and compressions forming the stress-couple of moment M at the other. From this free body, taking moments about the right-hand extremity, we have $M=\frac{1}{2}Px$, and hence the internal work for all the vertical laminæ of the left-hand half OB of the beam:

$$U\Big]_O^B=\int_0^B\frac{M^2dx}{2EI}=\frac{1}{2EI}\int_{x=0}^{x=l/2}\left(\frac{Px}{2}\right)^2dx=\frac{P^2}{8EI}\int_0^{l/2}x^2dx$$

$$=\frac{P^2}{8EI}\left(\frac{x^3}{3}\right)\Big]_0^{l/2}=\frac{P^2l^3}{192EI}. \quad . \quad (76)$$

From symmetry, we may double this for the internal work of the whole beam as due to flexure (that due to shear being neglected and that due to thrust being zero on account of the zero value of the thrust at all sections). That is, the total internal work for the beam is

$$U=\frac{P^2l^3}{96EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Now the deflection of point B below its original position in the horizontal line drawn through O and C, is the displacement of the point B in the direction of the external force P, and hence, by eq. (25),

$$y_1=\frac{dU}{dP}; \quad \text{i.e.,} \quad y_1=\frac{d}{dP}\left[\left(\frac{l^3}{96EI}\right)P^2\right];$$

P being now regarded as a variable. That is,

$$y_1=\left(\frac{l^3}{96EI}\right)2P; \quad \text{or} \quad y_1=\frac{Pl^3}{48EI}, \quad . \quad . \quad . \quad (78)$$

as we know by another method. (See page 254, Mech. of Eng.)

39. Influence of Shear in Preceding, if the Beam is a Steel I-Beam. If the beam in foregoing problem in a steel I-beam with narrow web the effect of shear in producing deflection may not be negligible. Considering the internal work of shear as well as that of the stress-couples, we have, since the total vertical shear in this case is $J=\frac{1}{2}P$ at all sections, for the internal work [see eqs. (74) and (77)],

$$U=\frac{P^2l^3}{96EI}+\frac{A}{2}\left(\frac{P}{2}\right)^2\cdot\frac{2}{F_wE_s}\int_0^{l/2}dx=\frac{l^3P^2}{96EI}+\frac{AlP^2}{8F_wE_s}, \quad . \quad (79)$$

in which F_w denotes the sectional area of the web alone while the coefficient A may be taken as unity. Therefore

$$y_1=\frac{dU}{dP}=\frac{Pl^3}{48EI}+\frac{Pl}{4F_wE_s}. \quad . \quad . \quad . \quad . \quad (80)$$

As a *numerical instance,* let us take the case of a 10-inch Cambria steel I-beam, weighing 25 lbs. per foot, and of a length of 10 feet; to be supported as in Fig. 16 and loaded at middle with $P=10{,}000$ lbs.; (the web placed in the usual vertical position).

For the cross-section of this beam we find from the Cambria Co.'s hand-book that I (about neutral axis perpendicular to the web) $=122.1$ in.^{4} while $F_w=7.85\times0.31=2.43$ in.^{2} Young's modulus may be taken as $E=30{,}000{,}000$ lbs./in.^{2}; and E_s, the modulus of elasticity for shearing, as $E_s=11{,}500{,}000$ lbs./in.^{2} With these values eq. (80) gives

$$y_1=0.0982+0.0107=0.1089 \text{ inch.}$$

The portion (viz., 0.0107) due to shearing action is seen to be about ten per cent of the whole.* (The weight of the beam itself has been neglected.) The effect of the shear would be much less marked with a beam of rectangular section; and also in the present case of the I-beam if the span were considerably increased.

* An interesting case of this nature is worked out by Professor C. J. Tilden in the *Engineering News* of Feb. 24, 1910, p. 228.

40. Example II. Deflection of Straight Prismatic Beam, Ends Supported, Load Uniform. (Fig. 17.) An (originally) straight homogeneous, prismatic beam rests on two supports

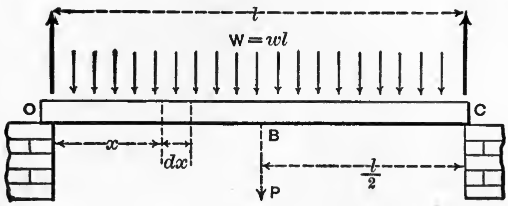

Fig. 17.

at its extremities at the same level, bearing a uniformly distributed load over its whole length of an amount $W = wl$ ($w =$ load per linear unit; l being the whole length). It is required to find the vertical deflection of the middle point B. Since there is no finite vertical load at this point let us imagine one ($= P$, say) to be placed there and finally reduce it to zero. We then have the reaction at each support $V = \frac{1}{2}(W+P)$. Now for any vertical lamina of thickness dx of the beam the moment of the stress-couple is

$$M = \tfrac{1}{2}(W+P)x - \frac{wx^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad (81)$$

x being measured from the left-hand support O. There is no thrust. Neglect the work of shear. To secure greater brevity let us use the form

$$y_1 = \int_O^C M\left(\frac{dM}{dP}\right)\frac{ds}{EI}, \text{ instead of } y_1 = \frac{d}{dP}\left[\int_O^C \frac{M^2 ds}{2EI}\right],$$

for the deflection of the point of application of the load P. Since the summation having to do with the dx's in the right-hand half of the beam is, from symmetry, equal to that in the left-hand half, we may double the expression applying to the left-hand half. Hence, noting that in this case $ds = dx$, and that, for any dx on OB, $M = \frac{1}{2}(W+P)x - \frac{wx^2}{2}$; and $\frac{dM}{dP} = \frac{x}{2}$; we have [see eq. (75)]:

$$y_1=\frac{2}{EI}\int_{x=0}^{x=l/2} M\left(\frac{dM}{dP}\right)dx$$

$$=\frac{2}{EI}\int_{x=0}^{x=l/2}\left[\tfrac{1}{2}(W+P)x-\frac{wx^2}{2}\right]\frac{x}{2}dx$$

$$=\frac{1}{EI}\left[\tfrac{1}{2}(W+P)\frac{x^3}{3}-\frac{w}{2}\cdot\frac{x^4}{4}\right]\Bigg|_{x=0}^{x=l/2};\text{ that is,}$$

$$=\frac{1}{EI}\left[\tfrac{1}{2}(W+P)\frac{l^3}{24}-\frac{W}{2}\cdot\frac{l^3}{64}\right]=\frac{l^3}{EI}\left[\frac{5W}{384}+\frac{P}{48}\right]. \quad (82)$$

This gives the deflection of B as due to the distributed load W and the concentrated load P at the middle. We have only to make P equal to zero to obtain the deflection B for the actual case of the distributed load alone, that is, $y_1=\frac{5}{384}\cdot\frac{Wl^3}{EI}$.

Note. The "dummy" force P having served its purpose in enabling us to get $\frac{dM}{dP}$, $=\frac{x}{2}$, we might have put P equal to zero in the expression for M above, before substituting in the integration, and it is also interesting to note that the value of $\frac{dM}{dP}$ is the same as the moment at any section due to a *load of unity* applied vertically at the point B (thus leading to "*Fraenkel's Formula.*" * Of course, also, by putting $W=$ zero in eq. (82) we may obtain the deflection due to P alone and thus check the result already obtained in eq. (78).

41. Example III. Deflection. Beam with Overhang. A continuous prismatic beam (homogeneous and originally straight) rests on two unyielding supports at the same level as shown in Fig. 18. The vertical force P is applied at D, midway between O and B, while the extremity of the overhanging portion carries a vertical load Q. The length $\overline{BC}$ equals $\overline{OB}$, $=2a$. Required the vertical deflection of the point D, neglecting the work of shear (that due to thrust being zero, as before). By eq. (25) this deflection $y_1=\frac{dU}{dP}$, where U

* See Mr. C. W. Hudson's paper on "Deflection of Beams of Variable Moment of Inertia," in Vol. LI (Dec., 1903) of the Transac. Am. Soc. Civ. Engineers, p. 1.

is the *total internal work* $= \int_O^C \frac{M^2 dx}{2EI}$. This summation will be separated into the three integrals (reckoning x for each segment in the way shown in Fig. 18) for OD, DB, and BC, respectively,

$$\int_0^a \frac{M^2 dx}{2EI}; \quad \int_0^a \frac{M^2 dx}{2EI}; \quad \text{and} \quad \int_0^{2a} \frac{M^2 dx}{2EI}.$$

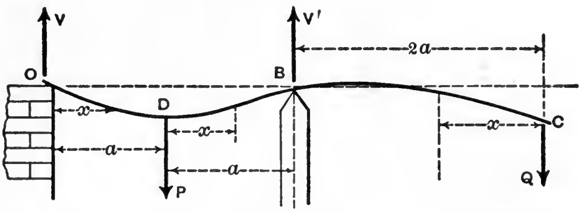

FIG. 18.

In this case we note that from ordinary statics the reaction at O is $V = \frac{P}{2} - Q$, and obtain the following expressions for M and its various derivatives in the three segments, viz.,

$$\text{For } OD, \quad M = Vx = \left(\frac{P}{2} - Q\right) x; \text{ and } \frac{dM}{dP} = \frac{x}{2};$$

for points on DB,

$$M = V(a+x) - Px, = \left(\frac{P}{2} - Q\right)(a+x) - Px; \quad \text{and} \quad \frac{dM}{dP} = \frac{a-x}{2};$$

while on CB, $M = Qx$ and $\frac{dM}{dP} = 0.$

Before substitution, however, we may write (differentiating under the integral sign),

$$_1 = \frac{1}{EI}\left[\int_O^D \left[M \cdot \frac{dM}{dP}\right] dx + \int_D^B \left[M \frac{dM}{dP}\right] dx + \int_C^B \left[M \frac{dM}{dP}\right] dx\right], \quad . \quad (83)$$

in which, wnen values are substituted from above relations for M, etc., there results

$$y_1=\frac{1}{EI}\left(\int_0^a\left[\frac{P}{2}-Q\right]\frac{x^2dx}{2}\right.$$

$$\left.+\int_0^a\left[\left(\frac{P}{2}-Q\right)(a+x)-Px\right]\left(\frac{a-x}{2}\right)dx+0\right). \quad (84)$$

Performing the integrations and reducing, we obtain

$$y_1=\left[\frac{P}{3}-Q\right]\frac{a^3}{2EI}; \quad . \quad . \quad . \quad . \quad . \quad (85)$$

In this equation, if we now make Q equal to zero, we have the same result as in eq. (78); while if P be made equal to zero we have the deflection (*upward;* note the negative sign) of the point D as due to the single load Q at C (in which case, of course, the extremity O must be "latched down," and the reaction V is downward).

It should be carefully noted that in the above solution P and Q are *independent loads;* that is, when P is conceived to vary Q remains constant; in other words, Q is not a function of P and hence (as above) $d(Qx)/dP=0$. But the *reactions,* V and V', *depend on both P and Q;* and neither V nor V' can be considered constant if either P or Q is to vary, but must be expressed in terms of the variable (P; above) before dM/dP can be obtained (on OD or DB). We note also that the external work done by these two reactions, when the loads P and Q are gradually applied, is *zero;* since the supports have smooth horizontal surfaces and the points of the beam in contact with them are thus free to slide (one or both) during the loading; thus justifying us in the application of eq. (25), or (19). [See note following eq. (19)].

In case the vertical deflection (y_2), of extremity C were desired, there being a vertical load Q at that point, we should have

$$y_2=\frac{dU}{dQ}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

the expansion of which would give us eq. (83) with P replaced by Q. In such case, of course, $d(Qx)/dQ$ is not zero, but $=x$. (Let the student work out the detail of finding y_2.)

42. Example IV. Equation to Curve of Straight Beam. Two End Supports. Single Eccentric Load. It is proposed to use our present methods for determining the equation of the elastic curve OD of the beam OC in Fig. 19 (homogeneous,

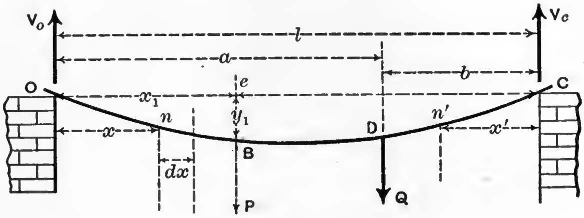

Fig. 19.

prismatic, and originally straight) resting on two end supports (smooth, horizontal surfaces) and loaded with a concentrated load Q not in the middle but at a distance a from the left-hand support greater than that, b, from the other support. (Denote $a+b$ by l.) We shall first find the vertical deflection of any point B (between O and D) whose distance from O is x_1, by the process just illustrated, involving the use of a "dummy" vertical force P applied at B. With such a force conceived to act, the reactions of the supports are evidently $V_0=\frac{b}{l}Q+\frac{l-x_1}{l}P$, and $V_c=\frac{a}{l}Q+\frac{x_1}{l}P$, respectively. The distance of any point n from O will be called x, and the moment of stress-couple at any point on OB is $M=\left[\frac{b}{l}Q+\left(\frac{l-x_1}{l}\right)P\right]x$; while $\frac{dM}{dP}=\frac{(l-x_1)}{l}x$. Again, for any section between B and D, we have

$$M=\left[\frac{b}{l}Q+\frac{l-x_1}{l}P\right]x-P(x-x_1); \text{ and } \frac{dM}{dP}=\left(\frac{l-x_1}{l}\right)x-(x-x_1).$$

For cross-sections on portion DC it will be more convenient to reckon x (calling it now x') from the support C; whence, for any section between D and C,

$$M=\left[\frac{a}{l}Q+\frac{x_1}{l}\cdot P\right]x'; \text{ and } \frac{dM}{dP}=\frac{x_1}{l}x'.$$

Now the deflection of the point B is $y_1=\frac{1}{EI}\int M\left(\frac{dM}{dP}\right)dx$, the summation extending over all the dx's (or dx''s) of the axis of the beam and can be written equal to the sum of three integrals [applying to the segments OB, BD, and DC, respectively], viz.:

$$\frac{1}{EI}\left[\int_{x=0}^{x=x_1} M\left(\frac{dM}{dP}\right)dx+\int_{x=x_1}^{x=a} M\left(\frac{dM}{dP}\right)dx+\int_{x'=0}^{x'=b} M\left(\frac{dM}{dP}\right)dx'\right],$$

or for brevity,

$$y_1=\frac{1}{EI}[Z_1+Z_2+Z_3]. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (87)$$

Before substituting details in eq. (87) we may put $P=0$ in each expression for M, since in this way the final result is the same as if the P had been retained and had not been placed equal to zero until the final expression had been reached; but each $\frac{dM}{dP}$, of course, is not equal to zero. In detail, therefore, bearing in mind these substitutions, we obtain, successively,

$$Z_1=\int_0^{x_1}\left(\frac{bQ}{l}x\right)\frac{(l-x_1)x}{l}dx=\frac{bQ}{l^2}(l-x_1)\int_0^{x_1} x^2dx=\frac{bQ}{l^2}(l-x_1)\frac{x_1^3}{3};$$

$$Z_2=\int_{x_1}^{a}\frac{bQ}{l}x\left[\frac{(l-x_1)x}{l}-(x-x_1)\right]dx$$

$$=\frac{bQ}{l}\int_{x_1}^{a}\left[\frac{(l-x_1)x^2dx}{l}-(x^2-x_1x)dx\right];$$

that is,

$$Z_2=\frac{bQ}{l}\left[\frac{(l-x_1)}{3l}[a^3-x_1^3]-\left(\frac{a^3-x_1^3}{3}-\frac{x_1(a^2-x_1^2)}{2}\right)\right];$$

while

$$Z_3=\int_{x'=0}^{x'=b}\frac{aQ}{l}x'\cdot\frac{x_1}{l}\cdot x'dx'=\frac{ax_1Q}{l^2}\int_0^b x'^2dx'=\frac{ax_1Q}{l^2}\cdot\frac{b^3}{3}.$$

Since B is arbitrary in position we may regard it as any point whatever on the elastic curve OD; and hence x_1 and y_1 become abscissa and ordinate of any such point. Therefore, substituting the values of Z_1, Z_2, and Z_3 just found, in eq. (87),

and with y for y_1 and x for x_1 (for simplicity), we have, as the equation to the elastic curve OD, O being the origin,

$$\frac{lEI}{Q}y=\frac{b}{l}(l-x)\frac{x^3}{3}+b\left[\frac{(l-x)}{3l}(a^3-x^3)-\left(\frac{a^3-x^3}{3}-\frac{x(a^2-x^2)}{2}\right)\right]+\frac{ab^3x}{3l}.$$

Or, after reduction,

$$\frac{lEI}{Q}y=\left[\frac{abl}{3}-\frac{a^2b}{6}\right]x-\frac{b}{6}x^3, \quad . \quad . \quad . \quad . \quad (88)$$

as the equation to elastic curve OD.

The abscissa of the lowest point of this curve (note that OD is $>DC$; i.e., $a>b$) which will thus locate the point of maximum deflection, may be obtained by writing

$$\frac{dy}{dx}=0;\ \text{i.e., by putting}\ \frac{Q}{lEI}\left[\frac{abl}{3}-\frac{a^2b}{6}-\frac{bx^2}{2}\right]=0;$$

which being solved, gives for the required abscissa x, $=x_m$,

$$x_m=\sqrt{\frac{a}{3}[2l-a]};$$

which can also be written

$$x_m=\sqrt{\tfrac{1}{3}(l^2-b^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

This value for x_m being substituted in eq. (88) gives for maximum deflection,

$$y_m=\frac{Qba}{9EIl}[2l-a]\sqrt{\frac{a}{3}(2l-a)}, \quad . \quad . \quad . \quad . \quad (90)$$

or, in another form,

$$y_m=\frac{Qb}{9EIl}(l^2-b^2)\sqrt{\tfrac{1}{3}(l^2-b^2)}. \quad . \quad . \quad . \quad . \quad (91)$$

(These results for x_m and y_m are seen to check with those on pp. 258, 494, and 506 of M. of E.)

43. Examples for Practice. Straight Prismatic Beams. The following involve horizontal straight beams of constant moment of inertia:

Example V. For the cantilever beam shown at (A) in Fig. 20, find the vertical displacement (i.e., deflection) of the extremity N, the load W being uniformly distributed over the

whole length; w is the load per running foot and l the length; I constant.

$$\tfrac{1}{8} \cdot \frac{Wl^3}{EI}. \qquad \textit{Ans.}$$

Example VI. The cantilever beam at (B) in Fig. 20 is built in, horizontally, at the left-hand extremity and carries a *variably* distributed load, W; the rate of distribution, w, is proportional to the distance x, from the free end, N. Find the vertical deflection of point N

$$\tfrac{1}{15} \cdot \frac{Wl^3}{EI}. \qquad \textit{Ans.}$$

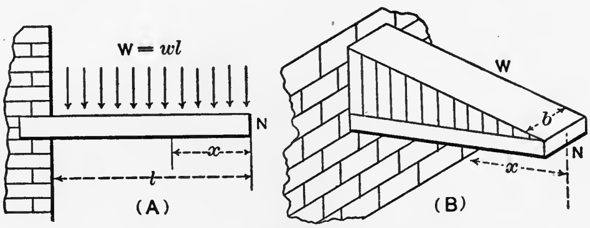

Fig. 20.

Example VII. Find the vertical displacement of *any* point on the elastic curve of the beam at (A), Fig. 20, in terms of the horizontal distance x_1, from extremity N.

$$y_1 = \frac{W}{24EIl}[3l^4 - 4l^3x_1 + x_1^4]. \qquad \textit{Ans.}$$

Example VIII. Find the vertical displacement of any point on the elastic curve of the beam at (B), Fig. 20, in terms of the distance x_1 from the extremity N.

$$y_1 = \frac{W}{60EIl^2}\left[4l^5 - 5l^4x_1 + x_1^5\right]. \qquad \textit{Ans.}$$

44. Horizontal Straight Beam of Variable Moment of Inertia (i.e., a Non-Prismatic Beam). Resting on Two Terminal Supports. As a case presenting this feature let us take the beam shown in Fig. 21. All its sections are rectangular and the height is constant throughout, $=h$. The two extreme quarters, however, are wedge-shaped, the width u

being proportional to the distance x from the end, while the middle half DA is prismatic, of constant width b. A single concentrated load, P_1, being placed at the middle point B, it is required to find the vertical displacement of that point; or y_1. From eq. (25),

$$y_1 = \frac{dU}{dP_1} = \int_0^C \frac{M}{EI} \cdot \frac{dM}{dP_1} \cdot dx;$$

that is, E being constant,

$$y_1 = \frac{2}{E}\int_0^B \frac{M}{I} \cdot \frac{dM}{dP_1} dx = \frac{2}{E}\left[\int_0^A \frac{M}{I} \cdot \frac{dM}{dP_1} dx + \int_A^B \frac{M}{I} \cdot \frac{dM}{dP_1} dx\right]. \quad (92)$$

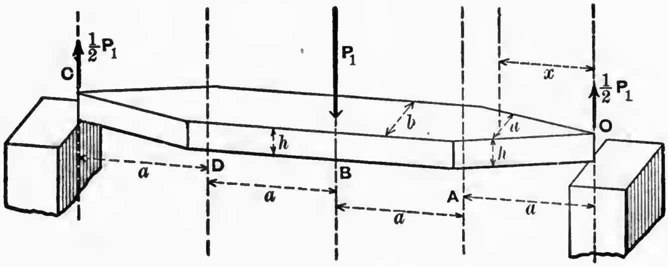

FIG. 21.

With x measured from support O, we have, for the dx's along AB, I being constant, $=I_0$,

$$M = \frac{P_1}{2}x; \quad \frac{dM}{dP_1} = \frac{x}{2}; \quad \text{and} \quad I = \frac{bh^3}{12}, = I_0;$$

while along OA (I variable),

$$M = \frac{P_1}{2}x; \quad \frac{dM}{dP_1} = \frac{x}{2}; \quad \text{and} \quad I, = \frac{uh^3}{12} = \frac{x}{a} \cdot \frac{bh^3}{12} = \frac{x}{a} \cdot I_0.$$

Hence eq. (92) becomes

$$y_1 = \frac{2}{E}\left[\int_0^a \frac{a}{xI_0} \cdot \frac{P_1 x}{2} \cdot \frac{x}{2} \cdot dx + \int_a^{2a} \frac{1}{I_0} \cdot \frac{P_1 x}{2} \cdot \frac{x}{2} \cdot dx\right]$$

$$= \frac{P_1}{2EI_0}\left[a\int_0^a x dx + \int_a^{2a} x^2 dx\right] = \frac{17}{12} \cdot \frac{P_1 a^3}{EI_0} = \frac{17P_1 a^3}{Ebh^3}. \quad (93)$$

45. Horizontal Non-Prismatic Beam. I Constant throughout Each of Several Portions of the Length. (Fig. 22.) In the case here shown the cross-section has different moments

of inertia (I) in the different portions OA, AB, etc., of the length (an I-beam with strengthened flanges, for instance), but the value of I is constant through any one such portion. With a symmetrical arrangement as indicated in Fig. 22, and with I_1 denoting the moment of inertia along AO and CE, I_2 that along AB and DE, and I_3 along BH and HD, we obtain y_1 of the middle point H by using a relation like that in eq. (92) but with three terms in the bracket instead of two; and in the three integrations concerned a different, but constant, I would occur in each. (See foot-note in § 41.)

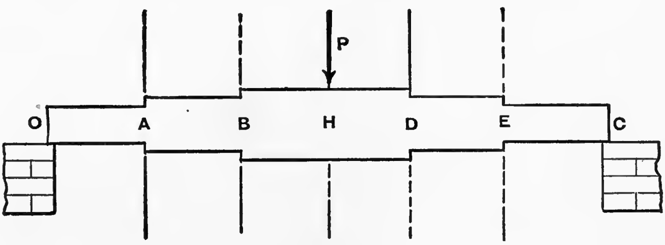

FIG. 22.

46. Straight Horizontal Beams on More than Two Supports (Statically Indeterminate). If the supported points of the beam are all free to slip along smooth fixed surfaces (except that *one* point must be fixed) and there are more than two points supported, the beam is *statically indeterminate;* i.e. the reactions (and consequently the internal stresses) cannot be determined by simple statics. Our present methods, however (Castigliano's Theorem and the Theorem of Least Work), enable solutions to be obtained by very simple and straightforward steps. A beam on more than two supports is called a "continuous girder."

It must be recalled that, in accordance with Castigliano's Theorem (§ 21), if a beam is supported at two points (one fixed; the other free to slide on a smooth fixed surface) the displacement y_1 of the point of application of some one load P_1, considered as a variable, in the direction of pointing of this load, will be given by $y_1 = \frac{dU}{dP_1}$, *provided* each of the other external forces either is independent of P_1, or, on account of being the

reaction of a smooth fixed support, *does no external work* when the beam is loaded.

If, therefore, a beam has two points supported and is to carry several loads, P, Q, etc., and we desire that the point of application of some one of them, say, Q, shall, during the gradual loading of the beam, execute no movement whatever in the direction of Q; in other words, that its displacement shall be *zero*; we have simply to write $\frac{dU}{dQ}=$ zero and solve the equation for Q. That is, if the point of application of Q is prevented from moving in the direction of Q, during the loading of the beam, by bearing against a fixed smooth surface at right angles to the direction of Q, the reaction of that surface will be Q, and must have the value computed from $\frac{dU}{dQ}=0$.

Hence if, before loading, the point of application of Q be chosen as a *third* point of support for the beam (with a fixed bearing surface provided at right angles to Q) the value of the reaction after loading may be found as indicated.

It is evident that the same reasoning applies to any elastic structure (without initial stresses) as well as to a beam, and also applies when there are *more* than two points of the beam or structure already provided with supports (*of the nature indicated above*); but, in any case, all the external forces which are dependent on the unknown Q *must be expressed as functions of* Q before the derivative in question is taken; then $\frac{dU}{dQ}=0$ will give Q in terms of other external forces.

47. Continuous Girder. Case I. (Fig. 23.) A straight, prismatic, homogeneous beam $C'OC$ is placed on three unyielding equidistant supports, at the same level. The two spans $C'O$ and OC, equal in length, each length being $l=a+b$, are loaded symmetrically with two loads, each $=P$; one in each span, at distance a from the middle support.

This beam is a " continuous girder " for which at the outset the reactions of the three supports are unknown. The reactions at C and C' are equal, from symmetry, each $=V$; that at support O is V'. They cannot be found by ordinary statics and it is now proposed to determine them by the principle announced in the last paragraph.

Selecting V as the reaction to be found by the method of § 46 we are to put $\frac{dU}{dV}=0$. For the present beam there is no thrust, and the work of shear will be neglected; so that, by eqs. (74) and (75), § 38, putting dx for ds,

$$\frac{dU}{dV}=\frac{d}{dV}\left[\int\frac{M^2ds}{2EI}\right]=\frac{1}{EI}\int M\cdot\frac{dM}{dV}\cdot dx=0, \quad . \quad . \quad (94)$$

extended over the whole beam. The moments (M) of the stress-couples along the beam will all be obtained in terms of the reaction V, at C (regarded as a variable), and of the known (constant) load P. On account of symmetry it will only be necessary to deal with the internal work, U, of the right-hand

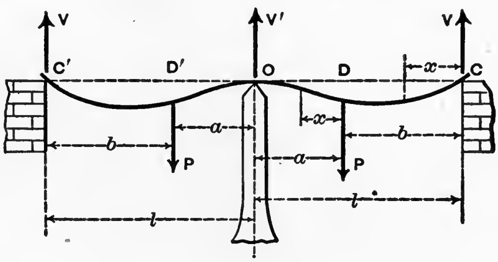

FIG. 23.

half, OC', of the beam; the expression involving which can then be doubled, for the total internal work.

For any dx on the portion DC (the abscissa x being measured as in the figure) we have

$$M=Vx; \quad \text{and} \quad \frac{dM}{dV}=x;$$

while for any dx on OD (and for these dx's we reckon x from point D),

$$M=V(b+x)-Px; \quad \text{and} \quad \frac{dM}{dV}=b+x.$$

Hence, substituting in eq. (94), we obtain

$$\frac{2}{EI}\left[\int_0^b Vx^2dx+\int_0^a(V[b+x]-Px)(b+x)dx\right]=0. \quad . \quad (95)$$

The integration being performed and the limits inserted, there results

$$V=\tfrac{1}{2}\cdot\frac{(3ba^2+2a^3)P}{b^3+3b^2a+3ba^2+a^3}. \quad . \quad . \quad . \quad . \quad (96)$$

(When once V is found the other reactions easily follow from ordinary statics) (viz., at C', V; and at O, $2P-2V$).

In the particular case where $b=a$ this gives

$$V=\tfrac{5}{16}P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

and hence $$V'=\tfrac{11}{8}P.$$

48. Continuous Girder. Case II. (Fig. 24.) Straight, prismatic, homogeneous beam on three equidistant and unyielding supports, O, B, and C, at same level. There is but one

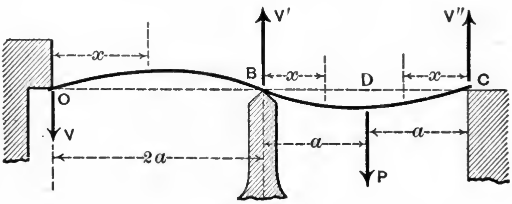

FIG. 24.

load, viz., P, in middle of left-hand span. Evidently the support at O must be placed above the beam and the reaction, V, at that extremity must point downward, since there is no load between O and B. V will be selected as the variable. The beam must be divided into three portions OB, BD, and DC for summation purposes, but first the other two reactions, V' and V'', since they depend on V, *must be expressed as functions of* V. This is done by ordinary statics, the whole beam being the "free body"; first taking moments about C, and then about B; whence

$$V'=2V+\tfrac{1}{2}P \quad \text{and} \quad V''=\tfrac{1}{2}P-V. \quad . \quad . \quad . \quad (98)$$

Reckoning x as shown in figure, for the three respective portions of beam, we have

on OB: $M=Vx$ and $\dfrac{dM}{dV}=x$;

on BD: $M=V(2a+x)-V'x, \; =V(2a+x)-(2V+\frac{1}{2}P)x;$

$$\text{i.e.,} \qquad M=V(2a-x)-\tfrac{1}{2}Px; \quad \text{and} \quad \frac{dM}{dV}=2a-x;$$

$$\text{on } DC: \quad M=V''x=(\tfrac{1}{2}P-V)x; \quad \text{and} \quad \frac{dM}{dV}=-x.$$

Hence, substituting in eq. (94),

$$\frac{1}{EI}\left[\int_0^{2a} Vx^2dx+\int_0^a [V(2a-x)-\tfrac{1}{2}Px](2a-x)dx \right.$$
$$\left. +\int_0^a (V-\tfrac{1}{2}P)x^2dx\right]=0,$$

whence, solving,

$$V=\tfrac{3}{32}P; \quad \therefore \; V'=\tfrac{22}{32} \quad \text{and} \quad V''\tfrac{13}{32}=P.$$

49. Continuous Girder. Case III. Non-Prismatic Beam. The straight homogeneous beam of Fig. 25 is supported at three equidistant points at same level. Supports firm. Two equal loads, each $=P$, one in middle of each span.

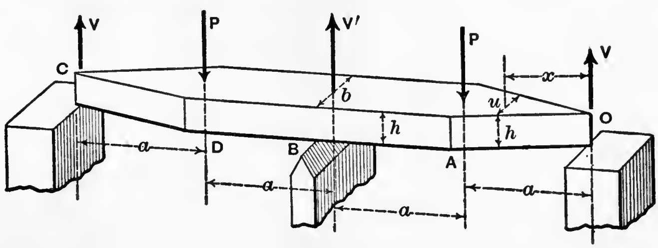

FIG. 25.

The middle half DA is prismatic, having a rectangular section with constant moment of inertia, I_0, $=bh^3\div 12$. The extreme quarters, however, CD and AO, have rectangular sections of constant height h and *variable* width u, u being proportional to distance x from O, or C; i.e., $u=(x\div a)b$, so that the moment of inertia along these portions of the beam is variable and is $I=uh^3\div 12=(x\div a)I_0$.

The unknown reactions being V, V', and V, if V be chosen as the variable its value is found by placing $\frac{2}{E}\int_O^B \frac{M}{I}\cdot\frac{dM}{dV}dx=\mathbf{0}$;

the work of shear being neglected and *advantage being taken of symmetry.*

On OA, $\quad M=Vx; \quad \frac{dM}{dV}=x;$ and $\quad I=\frac{x}{a}I_0.$

On AB, $\quad M=Vx-P(x-a); \quad \frac{dM}{dV}=x;$ and $\quad I=I_0.$

Hence

$$\frac{2}{EI_0}\left[\int_0^a \frac{aVx}{x}\,.\,xdx+\int_a^{2a}(Vx-P[x-a])xdx\right]=0.$$

$$\therefore \frac{Va^3}{2}+\frac{7Va^3}{3}-\frac{7Pa^3}{3}+\frac{3Pa^3}{2}=0, \quad . \quad . \quad . \quad (98a)$$

and $$\therefore V=\tfrac{5}{17}P; \quad \text{and} \quad V'=\tfrac{24}{17}P. \quad . \quad . \quad . \quad . \quad (98b)$$

50. Continuous Girder. Case IV. Symmetrically Placed, with Uniform Loading. (Fig. 26.) As in the preceding cases (except Case III), the beam is *prismatic* (I constant along whole length), with two equal spans; supports on a

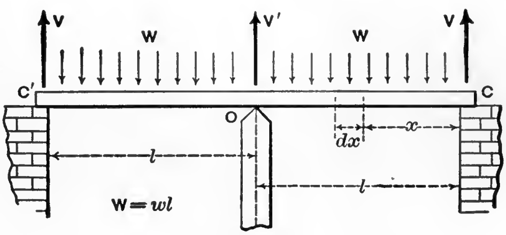

FIG. 26.

level; but the load is uniformly distributed over the whole length, at rate of w lbs. per foot. (Fig. 26.) The load on one span is $W=wl$, and the reactions are V, V' and V. As before, taking V as a variable, and doubling the result obtained from a consideration of the dx's along one span OC, or half the length of beam, we are to put

$$\frac{2}{EI}\int_C^O M\left(\frac{dM}{dV}\right)dx=0.$$

The M for any dx is the same function of x for all points of CO; i.e., from the free body of length x, we have along CO,

$$M = Vx - wx \cdot \frac{x}{2} = Vx - \frac{wx^2}{2}; \quad \text{and} \quad \frac{dM}{dV} = x$$

$$\therefore \frac{2}{EI}\int_0^l \left(Vx - \frac{wx^2}{2}\right) x dx = 0; \quad \text{i.e.,} \quad V\int_0^l x^2 dx - \frac{w}{2}\int_0^l x^3 dx = 0.$$

$$\therefore \frac{Vl^3}{3} - \frac{wl^4}{8} = 0; \quad \text{or} \quad V = \tfrac{3}{8}wl = \tfrac{3}{8}W, \quad \text{and} \quad V' = \tfrac{10}{8}W. \quad . \quad (98c)$$

51. Continuous Girder. Case V. (Fig. 27.) Beam prismatic, and loaded uniformly over whole length. Three points of support at same level. Length of right span double that of

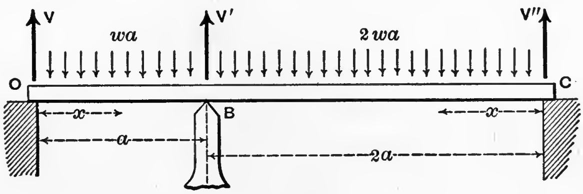

FIG. 27.

the left. Rate of loading, w, the same throughout the whole length, which is $3a$. E and I being constant, we may write,

$$\int_O^C M\left(\frac{dM}{dV}\right) dx = 0$$

(omitting the outside factor $1/EI$), if V be taken as a variable; the other reactions being V' and V''. Measuring x for the dx's on OB, and BC, as shown in the figure, respectively, we have for points on OB:

$$M = Vx - \frac{wx^2}{2} \quad \text{and} \quad \frac{dM}{dV} = x.$$

Taking a free body of length x measured from C, we obtain, for points on span BC, $M = V''x - wx^2$; but since a variation in V would involve a variation of V'' also, we must express V'' as a function of V before proceeding further. This is done, with the *whole beam* as a free body, putting by $\Sigma(\text{moms.}) = 0$, about B; whence

$$V'' = \frac{V}{2} + \frac{3wa}{4},$$

and

$$\therefore \quad M \text{ on } BC=\left(\frac{V}{2}+\frac{3wa}{4}\right)x-\frac{wx^2}{2}; \quad \text{and} \quad \frac{dM}{dV}=\frac{x}{2}.$$

Hence, substituting above,

$$\int_0^a\left(Vx-\frac{wx^2}{2}\right)xdx+\int_0^{2a}\left[\left(\frac{V}{2}+\frac{3wa}{4}\right)x-\frac{wx^2}{2}\right]\frac{x}{2}dx=0,$$

i.e.,

$$\frac{Va^3}{3}-\frac{wa^4}{8}+\frac{2Va^3}{3}+wa^4-wa^4=0.$$

$$\therefore \quad V=\frac{wa}{8}; \quad V'=\tfrac{33}{16}wa; \quad \text{and} \quad V''=\tfrac{13}{16}wa. \quad . \quad . \quad (99)$$

The shear and moment can now be computed for any cross-section of the beam.

52. Derivative of Internal Work, U, with Respect to M_1, the Moment of the Stress-Couple, at Any Section of a Beam in Flexure, when Statically Indeterminate. At any cross-section (1) of a beam, straight or curved, if we imagine the connection of the portions of the beam meeting at this section to consist of a hinge joint at the center of gravity of the cross-section, and two short bars (one on each side of the hinge, and at a proper distance from it) parallel to the axis of the beam at that point, and a distance a apart; then the pressure on the hinge joint will be the resultant of the thrust, T_1, and shear, J_1, at that section of the beam, while the stresses P_1 and P_1' in the two bars will be equal to each other, and the product P_1a is the moment, M_1, of the stress-couple.

Now if the beam is so supported as to be *statically indeterminate* and section 1 is situated between two supports the two short bars above mentioned might be omitted and the beam would still remain in place. In other words, each of the two bars is *redundant;* and we therefore write

$$\frac{dU}{dP_1}=0, \quad \text{and} \quad \frac{dU}{dP'_1}=0;$$

that is, dividing by a, and adding,

$$\frac{dU}{d(P_1a)}+\frac{dU}{d(P_1'a)}=0. \quad . \quad . \quad . \quad . \quad . \quad (100)$$

But this is simply the complete derivative of U, the internal work of the whole beam with respect to the moment, M_1, at cross-section 1. (The internal work of the two short bars themselves is considered zero; in other words, they are considered to be infinitesimal in length.) We may therefore write

$$\frac{dU}{dM_1}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (101)$$

This relation is available for determining the moment of couple M_1, at any given section (1) of a *statically indeterminate* beam, straight or curved (in the latter case the radius of curvature must be relatively large.)

53. Continuous Girders. Theorem of Three Moments. Although Castigliano's Theorem may still be used for determining the support reactions in case a continuous girder rests on *more* than three supports, the detail is very burdensome and tedious. The relation just proved in § 52, however, may be brought into play with great advantage in such cases, the treatment being very much simplified thereby. The beam being assumed to be homogeneous, straight, and prismatic (constant I), and all the supports to be at the same level and unyielding, a general relation based on eq. (101) will first be proved, known as Clapeyron's **Theorem of Three Moments,** established as follows:

In Fig. 28 we have a straight continuous beam with three points of its axis, viz., 0, 1, and 2 (0 and 2 being at the ends) supported on fixed supports at same level (two of these points are free to slide on smooth horizontal surfaces). Various vertical loads, P, P, etc., act on the beam; some in each span; of arbitrary values. Two *couples* are also applied to the beam, one at each end, of moments M_0 and M_2, respectively; that at the right being clockwise; that at the left, counterclockwise. These couples are virtually *arbitrary "loads,"* like the P's, being independent of them and of each other. It is plain that whatever value might be assigned to any one of these "loads," the beam would still be completely supported. Evidently the reactions V_0, V_1, and V_2, at the points of support, are each vertical.

For any set of given values for the "*loads*" there will be a corresponding value, M_1, of moment of (internal) stress-couple at cross-section 1, over the intermediate support at 1. The beam being *statically indeterminate*, M_1 may be found by the use of eq. (101); after rewriting it (neglecting work of shear) thus:

$$\frac{dU}{dM_1}=\frac{d}{dM_1}\left[\int_0^2 \frac{M^2 ds}{2EI}\right]=\frac{1}{EI}\int_0^2 M\left(\frac{dM}{dM_1}\right)dx=0. \quad . \quad (102)$$

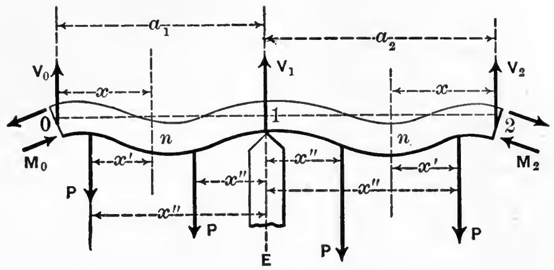

FIG. 28.

For sections on span 01 x is reckoned from 0; for those on span 02, from 2. For section n on 01, taking On as "free body" and n as center of moments, we have, for the moment M,

$$M=M_0-V_0x+\sum_n^0(Px'), \quad . \quad . \quad . \quad . \quad (103)$$

where $\sum_n^0(Px')$ denotes the sum of the moments, about n, of all the loads P *situated between n and* 0. But V_0 is a function of M_1, obtained by considering 01 as free body and taking moments about point 1, viz.:

$$V_0=\frac{1}{a_1}\left[\sum_1^0(Px'')+M_0-M_1\right]. \quad . \quad . \quad . \quad (104)$$

Here $\sum_1^0(Px'')$ denotes the sum of the moments, about point 1, of all the loads P between 1 and 0.

Now note that if the portion 01 of the beam were cut apart from the remainder and set up horizontally on two supports at its extremities 0 and 1, and loaded with the same loads P as before in same positions (but *without* the couples acting at 0

and 1) the reaction at the support 0 would be $R_0 = \frac{1}{a_1}\sum_1^0 (Px'')$, and the moment of couple at any section n would be

$$M_n = R_0 x - \sum_n^0 (Px'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

This ideal moment, M_n, which *would be* occasioned at any section n if the portion 01 were supported at its extremities as a "*simple beam*" [bearing the same loads P, but without the end couples] may be called the "*normal moment*" (as on p. 494, M. of E.).

As a result of substituting the expression for R_0 in eq. (105) and combination with eqs. (103) and (104) we readily derive for a point n in the *left-hand* span $0 \ldots 1$,

$$M = M_0 - M_n - (M_0 - M_1)\frac{x}{a_1}, \quad . \quad . \quad . \quad (106)$$

giving M as a function of the *one* variable M_1; whence

$$\frac{dM}{dM_1} = \frac{x}{a_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Similarly, there are obtained, for any point n in the *right span* $1 \ldots 2$,

$$M = M_2 - M_n - (M_2 - M_1)\frac{x}{a_2}, \quad . \quad . \quad . \quad (108)$$

and

$$\frac{dM}{dM_1} = \frac{x}{a_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

Substituting in eq. (102), which may be written

$$\int_{x=0}^{x=a_1} M \cdot \frac{dM}{dM_1} \cdot dx + \int_{x=0}^{x=a_2} M \cdot \frac{dM}{dM_1} \cdot dx = 0, . \quad (110)$$

we obtain

$$\left.\begin{aligned} &\frac{1}{a_1}\int_0^{a_1}\left[M_0 x - M_n x - (M_0 - M_1)\frac{x^2}{a_1}\right]dx \\ +&\frac{1}{a_2}\int_0^{a_2}\left[M_2 x - M_n x - (M_2 - M_1)\frac{x^2}{a_2}\right]dx \end{aligned}\right\} = 0. \qquad (111)$$

Of the quantities a_1, a_2, M_0, M_1, M_2, and M_n, only M_n is a function of x in each span; this ideal moment, or "normal moment," having the meaning just given. Performing the

integrations, as far as it can be done in the general case, we have, after reduction,

$$M_0a_1+2M_1(a_1+a_2)+M_2a_2=\frac{6}{a_1}\int_0^{a_1} M_nxdx+\frac{6}{a_2}\int_0^{a_2} M_nxdx, \quad (112)$$

which is Clapeyron's **Theorem of Three Moments,** giving a relation between the three moments M_0, M_1, M_2, of the couples at the three points 0, 1, and 2; of a homogeneous, prismatic beam supporting a set of vertical loads P, P, etc.; the three points in question resting on three fixed supports at same level.

Note A. It must be *carefully noted* that, of the integrals in the right-hand member of (112), the first one deals with the *left-hand span* and its loads, x being measured from its *left* end, or point 0; while the second relates to the *right-hand span,* x being measured from its *right* end or point 2; and the ideal M_n must be a *positive* number.

Note B. The three moments, M_0, M_1, and M_2, have been used in such a way as to imply tension in the upper fibers of the beam, and results must be interpreted on this basis; that is, if a negative number is obtained for one of them it implies that at that section of the beam the upper fibers are in compression.

54. Values of the Two Integrals in the Three-Moment Theorem in Particular Cases. For brevity, the first integral (referring to loads in the left-hand span) in eq. (112) may be denoted by Z_1, and the second (right-hand span) by Z_2; and the theorem will then read

$$M_0a_1+2M_1(a_1+a_2)+M_2a_2=\frac{6}{a}\,.\,Z_1+\frac{6}{a_2}\,.\,Z_2. \quad . \quad (113)$$

If the loading on the left-hand span is a **uniformly distributed load,** W_1 covering the *whole span* (see Fig. 28) a_1 at rate w_1 lbs./lin.ft., the value of M_n at any point n in portion 01 [conceived to be a separate beam resting on two supports at its extremities] would be $\frac{W_1}{2}x-\frac{w_1x^2}{2}$; hence (with $W=w_1a_1$),

$$Z_1=\int_0^{a_1}\left[\frac{w_1a_1x}{2}-\frac{w_1x^2}{2}\right]xdx=\frac{w_1a_1^4}{24}=\frac{W_1a_1^3}{24}. \quad . \quad (114)$$

Similarly, if the right-hand span carries a load $W_2, =w_2a_2,$ covering the *whole span* a_2, we have

$$Z_2=\tfrac{1}{24}W_2a_2^3.$$

With a **single concentrated load** P in the *left-hand span* (Fig. 28) at a distance x'' from support 1, we have $M_n=\dfrac{Px''}{a_1}x$ for any point n on the left of P, but for any point n between P and support 1, $M_n=\dfrac{Px''}{a_1}x-P[x-(a_1-x'')]$; hence

$$Z_1=\int_0^{a_1-x''}\left(\frac{Px''x}{a_1}\right)xdx+\int_{a_1-x''}^{a_1}\left(\frac{P''x''x}{a_1}-P[x-(a_1-x'')]\right)xdx,$$

i.e.,

$$Z_1=\tfrac{1}{6}P(a_1-x'')(2a_1-x'')x''; \quad . \quad . \quad . \quad (115)$$

and similarly, for a single concentrated load P in the *right-hand span*, at distance x'' from support 1,

$$Z_2=\tfrac{1}{6}P(a_2-x'')(2a_2-x'')x''.$$

(If for a_1-x'' we write z'' (distance of P from support 0) Z_1 in (115) becomes

$$\tfrac{1}{6}P(a_1^2-z''^2)z''; \quad \text{and} \quad Z_2=\tfrac{1}{6}P(a_2^2-z''^2)z'',$$

for a load P in right-hand span, with $z''=a_2-x''$.)

For **any number of concentrated loads** in the left-hand span, Z_1 is the sum of a number of terms of the form given in the right-hand member of eq. (115), one term for each load, and containing the amount, P, of the load and the corresponding x''. A similar statement may be made for Z_2, as regards the right-hand span, if that span carries a number of concentrated loads; *remembering that the x'' is measured from support* 1 (see Fig. 28).

Of course, if there is **no load** on the left-hand span, Z_1 is *zero;* and similarly for the right-hand span, with no load on that span, $Z_2=0$.

For a **distributed load over any portion** of the left-hand span, the rate of distribution w being constant or variable, we may find the value of Z_1 by substituting wdx'' for P in eq. (115)

and summing (i.e., integrating) all such terms between the extremities of the distributed load; that is, by making x'' vary between proper limits. If w is variable it must first be expressed in terms of the *variable* x''. For example, if a load W_1 of *constant* rate w_1 is distributed over the part of the left-hand span which extends c_1 ft. toward the left from support 1 (so that $W_1 = w_1 c_1$), we have

$$Z_1 = \frac{1}{6}\int_{x''=0}^{x''=c_1} (w_1 dx'')(a_1 - x'')(2a_1 - x'')x''$$

$$= \frac{W_1 c_1 (2a_1 - c_1)^2}{24}; \quad . \quad (116)$$

and similarly if, in the *right-hand span*, a uniformly distributed load, W_2, of rate w_2, extends to the right over c_2 ft. *from the support* 1, we have

$$Z_2 = \frac{1}{24} W_2 c_2 (2a_2 - c_2)^2.$$

Again, if, in the left-hand span, the uniformly distributed load W_1, $= w_1 d_1$, of rate w_1, extends d_1 ft. toward the right *from the support* 0, by a similar process we obtain (now making x'' vary between the limits $a_1 - d_1$ and a_1),

$$Z_1 = \frac{W_1 d_1 (2a_1^2 - d_1^2)}{24}; \quad . \quad . \quad . \quad . \quad . \quad (117)$$

while a distributed load W_2, $= w_2 d_2$, at constant rate w_2, covering the portion of *right-hand span* extending toward the left d_2 ft. from support 2, calls for a value of $Z_2 = \frac{1}{24} W_2 d_2 (2a_2^2 - d_2^2)$.

If neither end of a distributed load adjoins a support, results for Z_1 and Z_2 are obtained by the same methods, by simply using proper limits for the variable x''; (and if w is variable it must first be expressed as a function of x'').

In conclusion it may be said that, for any kind of (vertical) loading whatever, on either span, the value of Z_1 (or Z_2), is made up of the sum of the separate terms that would apply to each separate part of the loading, concentrated or distributed, carried by the beam in the span in question.

55. Continuous Girder: (Prismatic) Resting on an Indefinite Number of Fixed Supports at the Same Level. In such a case it is convenient to let the moments in the sections just over

the various supports constitute a set of unknown quantities to be deterimned at the outset, use being made of the *Theorem of Three Moments*, just proved in § 53.

To apply this method we must note that the beam in Fig. 28 may be considered to be a portion (isolated as a "free body"), comprising *any two consecutive spans*, of a long continuous girder of several spans. At 0, a section has been made cutting close on the right of a support, M_0 being the moment of the couple acting in the section, while V_0 may represent the shear in the section; while 2 is a section made close on the left of a support (the second support from the other), M_2 being the moment of couple and V_2 representing the shear in the section. We have therefore only to apply this relation to as many lengths, of two consecutive spans each, as circumstances permit, in order to obtain as many equations as there are unknown quantities.

For example, if there are **five spans,** i.e., six points of support at same level (the beam simply resting on each support; not clamped or "built-in") the moments over the six points of supports are the quantities M_0 (over the extreme left support; and M_0 is either zero if the beam terminates there or there is an unloaded overhang; or is already known if there is a loaded overhang), M_1, M_2, M_3, M_4, and finally M_5 (which is either zero if there is no overhang, or an unloaded overhang; or is already a known quantity if there is a loaded overhang). Here the theorem may be applied four times: First to the two consecutive spans extending from support 0 to support 2; then from 1 to 3; then from 2 to 4; and finally from 3 to 5. The equations thus obtained are seen to be four in number, containing no unknown quantities except the four moments, M_1 to M_4 inclusive, which are then determined by ordinary algebra. With these moments known, all reactions, and the moment and shear at any section, are easily found by ordinary statics.

In case the **left-hand end of the beam is clamped,** or "built-in," horizontally, we may consider that the first actual span on the left is the right-hand span of two consecutive spans of which the left-hand span is indefinitely short ($a_1=0$) and carries no load; so that the theorem, being applied to

these two spans gives a result obtained by putting a_1 and Z_1 each equal to zero in eq. (113),

$$2M_1a_2+M_2a_2=(6\div a_2)Z_2. \quad . \quad . \quad . \quad . \quad (118)$$

Similarly, if the right-hand extremity of the girder is fixed, or built-in, horizontally, we consider the last actual span on the right to be the left-hand span of two consecutive spans of which the right-hand one is indefinitely short, and carrying no load; that is, we put a_2 and Z_2, each equal to zero in eq. (113), and obtain

$$M_0a_1+2M_1a_1=(6\div a_1)Z_1. \quad . \quad . \quad . \quad . \quad (119)$$

Of course in both of these special cases the built-in section of the beam is represented by the section at point 1 in Fig. 28; with corresponding meaning of the symbols in the above two equations.

56. Continuous Girders. Case VI. (Fig. 29.) This figure shows a continuous girder extending over three spans and with its extremities resting (not built-in) on the extreme supports,

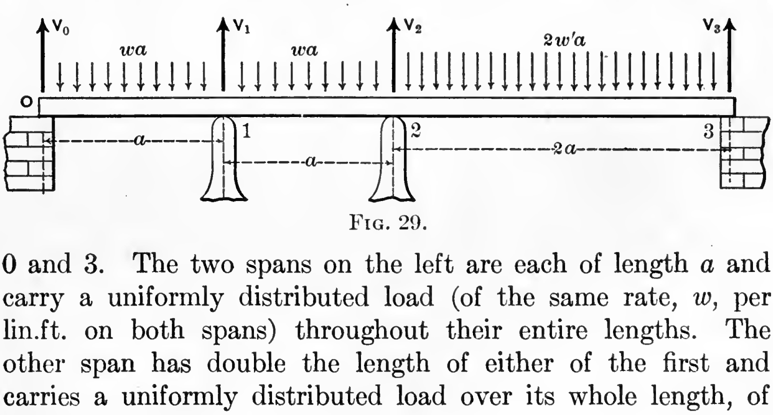

FIG. 29.

0 and 3. The two spans on the left are each of length a and carry a uniformly distributed load (of the same rate, w, per lin.ft. on both spans) throughout their entire lengths. The other span has double the length of either of the first and carries a uniformly distributed load over its whole length, of rate w'.

The girder is homogeneous and prismatic and the four supports are at same level, hence the Theorem of Three Moments is available [eqs. (112) and (113)]. Applying it first to the two consecutive spans on the left, viz., 0 . . . 1 and 1 . . . 2, we find [see eq. (114)], with $M_0=0$,

$$0+2M_1(a+a)+M_2a=\frac{6}{a}\cdot\frac{(wa)a^3}{24}+\frac{6}{a}\cdot\frac{(wa)a^3}{24}=\frac{wa^3}{2}. \quad (120)$$

Next applying it to the (consecutive) spans 1 . . . 2 and 2 . . . 3, we have, with $M_3=0$,

$$M_1a+2M_2(a+2a)+0=\frac{6}{a}\cdot\frac{(wa)a^3}{24}+\frac{6}{2a}\cdot\frac{(2w'a)(2a)^3}{24},$$

i.e., $$M_1a+6M_2a=\tfrac{1}{4}wa^3+2w'a^3. \quad . \quad . \quad . \quad . \quad (121)$$

By elimination between (120) and (121) we have

$$M_1=\tfrac{11}{92}wa^2-\tfrac{2}{23}w'a^2; \quad \text{and} \quad M_2=\tfrac{1}{46}wa^2+\tfrac{8}{23}w'a^2.$$

Assuming that w is greater than $\frac{8}{11}w'$, we note that both M_1 and M_2 are positive; that is, that the couples acting in the sections at 1 and 2 are of a character producing tension in the upper fibers at those sections (according to Note B of § 53). In other words, the curvature of the beam is convex on the upper side; in the way indicated at the points 0, 1, and 2, in Fig. 28.

The reactions V_0, V_1, etc., are now easily found by statics. For example, from the free body 0 . . . 1, reaching from support 0 to a cutting plane just short of support 1, we find acting at the right-hand end of the body a shear, J', and a couple (clockwise) whose moment is M_1, just obtained. Putting Σ(moms.)$=0$ about the neutral axis of this section for this body we find

$$J'\times 0-V_0a+wa\,.\,\frac{a}{2}-M_1=0,$$

and hence $$V_0=\tfrac{35}{92}wa+\tfrac{2}{23}w'a, \quad . \quad . \quad . \quad . \quad . \quad (122)$$

and similarly we may find the other three reactions, taking various "free bodies"; the whole beam being conveniently taken as such, for two of them.

CHAPTER V

Composite Systems; and Curved Beams

57. Straight Prismatic Beam Supported by Three Others. Symmetrical Design. In Fig. 30 we have a system of four horizontal straight beams, each prismatic and homogeneous; the upper one, A, resting on the other three, B, B', and B, underneath; and these latter, on unyielding supports, C' and C'. A is at right angles to the others, touching the middle of each. The middle of A touches B'; and its two extremities touch

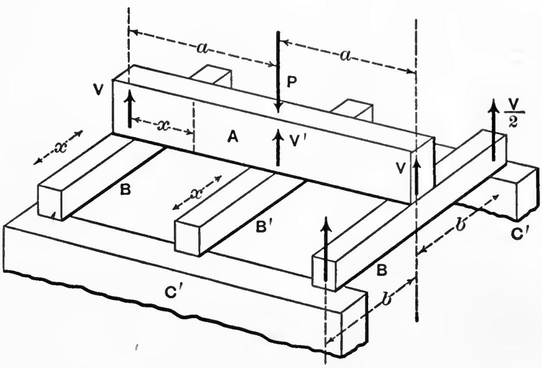

Fig. 30.

B and B. For A we have the values E and I; for each of the other beams, E' and I'. There is but one load, viz., P, vertical and applied at the middle of A. Weights of beams neglected. Before the system is loaded the beams are in *close contact* with each other and with the supports, *without strain;* Castigliano's Theorem is therefore applicable; and its corollary (Least Work).

The nine pressures between the beams and against the supports are all unknown in advance and cannot be determined by statics alone, but they can all be expressed in terms of one of them, say, V, the pressure under the left-hand extremity of beam A. That at the right extremity of A is also V; and that, V', under the middle of A is readily seen

to be equal to $P-2V$, for the equilibrium of A. Under each end of beams B, B, the pressure is $V/2$. Under each end of B' it is $V'/2$; that is, $(P-2V)/2$. The various half lengths are a and b; while the x of any point is measured as in Fig. 30.

If we consider the left end of beam A to bear on beam B by the interposition of an infinitely short vertical bar the stress in this bar will be the unknown V, and we may take this bar as the redundant bar of the system; and hence (by § 29) write $dU/dV=0$ (the internal work of the bar itself being, of course, zero). Here U must denote the total internal work of the six beams, each of which is in a state of flexure, but not under thrust. The work of shear will be neglected. Evidently the internal work U_A, of beam A, is double that of one of its halves, from symmetry (in this case). A similar statement applies to U_B', for B'; while for the two beams B and B the internal work U_B is four times that of the half length of one of them. It remains to fill in the detail of the relation

$$\frac{dU}{dV}=\frac{d}{dV}\left[\int\frac{M^2ds}{2EI}\right]=\sum\left[\frac{1}{EI}\int M\left(\frac{dM}{dV}\right)dx\right]=0. \quad (123)$$

For beam A,

$$M=Vx;\quad \frac{dM}{dV}=x;\quad \therefore\quad \frac{dU_A}{dV}=\frac{2}{EI}\int_0^a Vx\,.\,xdx=\frac{2Va^3}{3EI}.$$

For the two beams B, B,

$$M=\frac{Vx}{2};\quad \frac{dM}{dV}=\frac{x}{2};\quad \therefore\quad \frac{dU_B}{dV}=\frac{4}{E'I'}\int_0^b \frac{Vx}{2}\cdot\frac{x}{2}dx=\frac{Vb^3}{3E'I'}.$$

For beam B',

$$M=\frac{V'x}{2}=\left(\frac{P-2V}{2}\right)x;\quad \frac{dM}{dV}=-x;$$

and

$$\frac{dU_{B'}}{dV}=\frac{2}{E'I'}\int_0^b\frac{(P-2V)x}{2}(-x)dx=\frac{2Vb^3}{3E'I'}-\frac{Pb^3}{3E'I'}.$$

Substituting these values in eq. (123) we obtain

$$\frac{2Va^3}{3EI}+\frac{Vb^3}{3E'I'}-\frac{Pb^3}{3E'I'}=0;\quad \therefore\quad V=\frac{P}{2(E'I'\div EI)(a\div b)^3+3}.$$

If all the beams are of the *same material* and the *same moment of inertia* of cross-section, and $a=b$, this reduces to $V=\frac{1}{5}P$ and $V'=\frac{2}{5}P$.

Example I. Find the value of V for the case in Fig. 30 in case the weights of the beams are *not* neglected (weight per running foot being w for beam A, and w' for each of the others).

Example II. Instead of the concentrated load applied at the middle of beam A, apply a load W distributed uniformly over the whole length of beam A, in Fig. 30; first, neglecting the weights of the beams; then, considering those weights.

For cases resembling that of Fig. 30, but in which the arrangement is *not symmetrical,* the mode of treatment will be the same but evidently more detail will be involved.

58. Trussed-Beam (or Brake-Beam). In Fig. 31 at (A) is shown a straight, prismatic, homogeneous beam with its two extremities resting on two piers at the same level;

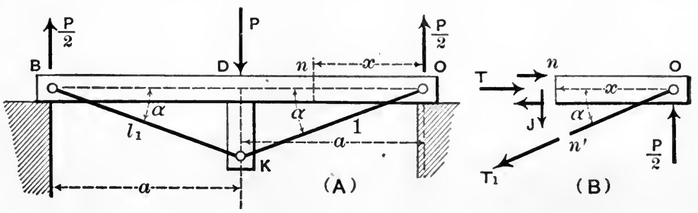

FIG. 31.

provided with a short vertical strut, DK, at its middle point, and also with two oblique tie-rods connecting the lower end of the strut with the two extremities of the beam. It is to be understood that these four members are accurately fitted and secured to each other, before loading, in the position shown, but not so as to create any stress in any member; in other words, the beam is still straight, and stresses will begin to be produced in *all* the members as soon as any load is placed upon the beam; that is, there are no "initial stresses."

In the present case the only load is the concentrated vertical load P, at the point D; when bearing which, the structure is assumed to be *at the same temperature as when the parts were put together.* It is required, by the method of "Least Work," to find: The tensile stress, T_1, induced in each tie-rod; the

compressive force produced in the strut, DK; the thrust, T, produced in the horizontal beam itself; and an expression from which we can compute the "bending moment," M, at any section of the beam. As regards the total internal work, U, of all the members, that of the strut (on account of its short length) will be neglected, that of each tie-rod will be an expression for the case of simple tension, viz., $\frac{C_1T_1^2}{2}$, while that for the beam will consist of a term $\frac{CT^2}{2}$ for the work of the thrust, and one, $\int\frac{M^2dx}{2EI}$, for the work of the "bending moments"; (that due to the shears being neglected).

Here C_1 denotes $\frac{l_1}{F_1E_1}$ (see § 20), where l_1 is the length and F_1 the sectional area of each tie-rod; E_1 being the modulus of elasticity. Also $C=l\div FE$, where l $(=2a)$ is the length, and F the sectional area, of the horizontal beam itself; and E the modulus of elasticity of its material. The angle DBK is denoted by α (see Fig. 31).

At (B) in Fig. ~~30~~ 31, is shown the free body nOn' with all the forces external to it, we have from simple statics (since $P/2$ is evidently the reaction of each support),

$$T=T_1\cos\alpha;\quad\text{and}\quad M=\frac{P}{2}x-T_1x\sin\alpha,$$

(so that $\frac{dM}{dT_1}=-x\,.\,\sin\alpha$); which are to be substituted in the expression for internal work of the whole structure, viz.:

$$U=2\left[\frac{C_1T_1^2}{2}\right]+\frac{CT^2}{2}+\frac{2}{EI}\int_0^D\frac{M^2dx}{2}. \quad . \quad . \quad . \quad (124)$$

By differentiating this last expression with respect to T_1 we have

$$\frac{dU}{dT_1}=2C_1T_1+CT\frac{dT}{dT_1}+\frac{2}{EI}\int_0^D M\left(\frac{dM}{dT_1}\right)dx, \quad . \quad (125)$$

which, after substitution of above relations, is to be put equal to zero, according to eq. (49). We thus progressively derive

$$2C_1T_1+CT_1(\cos\alpha)\cos\alpha$$
$$+\frac{2}{EI}\int_0^{x=a}\left(\frac{Px}{2}-T_1x\,.\sin\alpha\right)(-x\,.\sin\alpha)dx=0,$$

or,

$$T_1[2C_1+C\cos^2\alpha]-\frac{2\sin\alpha}{EI}\int_0^a\left[\frac{P}{2}x^2dx-T_1(\sin\alpha)x^2dx\right]=0;$$

that is,

$$T_1(2C_1+C\cos^2\alpha)-\frac{2\sin\alpha}{EI}\left[\frac{Pa^3}{6}-\frac{T_1a^3\sin\alpha}{3}\right]=0, \quad (126)$$

and finally, solving for T_1, the tension in either tie-rod, noting that a is only the *half-length* of the beam, we obtain

$$T_1=\frac{Pa^3\sin\alpha}{EI(6C_1+3C\cos^2\alpha)+2a^3\sin^2\alpha}. \quad . \quad . \quad (127)$$

T, and the thrust in DK, are now easily found, and the value of M at any point of beam; and finally the maximum unit stress (which should be below the elastic limit) in the outer fibers of the beam; as due to combined thrust and bending moment.

If all members of the system are of the same material they expand, or contract, together, in case the temperature changes after the parts are put together; but if the beam were of *timber*, for example, and the tie-rods *iron* or *steel* (as is often the case), the metal would expand so much more than the timber, for a rise of temperature, that the maximum moment, and stresses dependent on it, in the beam, might be much larger than if the temperature had remained at the original value (see next paragraph). Similarly, a fall of temperature might greatly alter the stresses.

59. Cambered Trussed-Beam; and Effect of Change in Temperature in an Ordinary Trussed Beam. If it is desired to give a "**camber**" or slight curvature (convex on upper side) to the beam of the foregoing article, before it is loaded, this may be done by making the vertical strut a little longer than is called for by the exact fitting of all the parts without strain.

In such a case, to find the stresses in the tie-rods, etc., after loading, use may be made of § 32, as follows:

In Fig. 32 the tie-rods BK and KO are shown in such a position that the distance from B to O is horizontal and equal to the length of the beam (between joint centers). The strut KD, of a length a little greater than for no camber, being placed vertically with lower extremity at K, its upper end D projects a small distance $\overline{D'D}$, $=\lambda'$, above the line joining B and O, and hence when the axis of the beam is placed so as to pass through B and K, with one extremity at B, the other extremity will be found at O'. If now we join KO' and let fall the perpendicular $O \ldots e$ upon the line KO', we (practically) determine the distance $\overline{O'e}$, $=\lambda_0$, as the amount by which the tie-rod

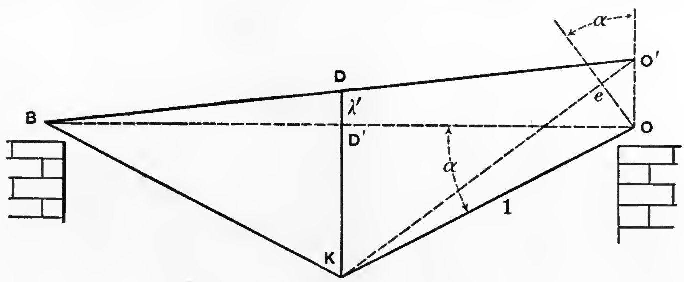

FIG. 32.

on the right is *too short* to connect the "joints" K and O'. This is called λ_0 in § 32, the bar in question being a redundant member. Hence the stress, T_1, in the tie-rod, after the parts are forced together and the structure loaded (at the same temperature), may be found from eq. (57) of § 32; that is, by putting $dU/dT_1=\lambda_0$, the stress T_1 being assumed to be tension. From Fig. 32 we have $\overline{O'e}=\overline{OO'}\,.\sin\alpha=2\overline{DD'}\,.\sin\alpha$; i.e., $\lambda_0=2\lambda'\sin\alpha$.

We may therefore find the stress T_1 in the tie-rod by putting λ_0 instead of zero in the right-hand member of eq. (126); thus obtaining

$$T_1=\frac{Pa^3\sin\alpha+3EI\lambda_0}{EI(6C_1+3C\cos^2\alpha)+2a^3\sin^2\alpha}. \quad . \quad . \quad (128)$$

If, with the *uncambered* trussed-beam, the temperature, t', after loading, is *lower* than that, t_0, when the parts were put

together, it becomes evident, from a figure drawn to represent the (straight) axes of the beam, the strut, and *one* of the tie-rods, as assembled (unstrained) at the new temperature t', that the other tie-rod is too short to fit into place by some amount, λ_0, T_1 may be found from eq. (128) by insertion therein of the value of λ_0. For example, if the coefficient of expansion or contraction for the *steel* tie-rods is η, while that of the *timber* beam is negligible in comparison, as also that of short strut, we have,

$$\lambda_0 = 2\eta(t' - t_0)l_1.$$

If the temperature t' of loading is *higher* than the original temperature t_0 of "assembling," with steel tie-rods and timber beam, we replace the λ_0 of eq. (128) by the (negative) quantity: $-2\eta(t' - t_0)l_1$; for instance, if $t' = 35°$, and $t_0 = 15°$, C., with $l_1 = 10$ ft. $= 120$ in. we have

$$\lambda_0 = -2(0.000011)(35 - 15)(120) = -0.053 \text{ inch.}$$

60. Other Cases of Trussed-Beams. Example I. Find the stress in one of the tie-rods of the trussed-beam in Fig. 31 in case the loading, instead of being concentrated in the middle D, is a uniformly distributed load, W, covering the whole length (2) of the beam at rate w lbs. per lin.ft. Disregard changes of temperature.

$$T_1 = \frac{5wa^4 \sin \alpha}{EI(24C_1 + 12C \cos^2 \alpha) + 8a^3 \sin^2 \alpha}. \quad \textit{Ans.}$$

Example II. In Fig. 31 suppose the single concentrated load P to be placed at a point in the right-hand half of the beam and at a distance $= c$ from support O. Find T_1, the stress in tie-rod OK.

Example III. Trussed-Beam with Two Struts and Three Tie-rods. Fig. 33 shows this case; the load, W, $= 3wa$, being uniformly distributed over the whole length, $3a$, of the beam; which is prismatic and homogeneous and is provided (at points dividing its length into thirds) with two short vertical struts, and three tie-rods 1, 2, and 3, as shown; tie-rod 2 being horizontal. The figure shows the notation as to dimensions and angles. The internal work of the struts will be neglected, as also the work of shear in the beam. Rods 1 and 3 are equal in

sectional area F_1 and length l_1 ($=a \sec \alpha$), their modulus of elasticity being E_1; corresponding quantities for rod 2 being F_2, E_2, and l_2 ($=a$); and for the beam, F, E, and l ($=3a$) (with I as moment of inertia of section).

Let $\dfrac{l_1}{F_1E_1}$ be denoted by C_1; $\dfrac{l_2}{F_2E_2}$ by C_2; and $\dfrac{3a}{FE}$ by C.

The four members, assumed to be fitted together closely, without strain, before loading, form an elastic system with one redundant bar. Bar (rod) 1 will be taken as the redundant

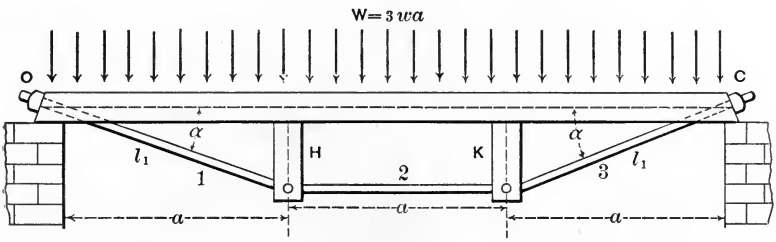

FIG. 33.

bar; and the stress in it, T_1, is to be determined, for the given loading; change of temperature being disregarded. Hence by § 29 we write $dU/dT_1=0$. Let $T=$the thrust in the beam (evidently the same at all sections), and M the (variable) "moment" in any section; and T_2, the tensile stress in rod 2. In the present case we have, therefore,

$$\frac{dU}{dT_1}=2C_1T_1+C_2T_2\frac{dT_2}{dT_1}+CT\frac{dT}{dT_1}$$
$$+\frac{2}{EI}\int_O^H M\cdot\frac{dM}{dT_1}\cdot dx+\frac{1}{EI}\int_H^K M\cdot\frac{dM}{dT_1}dx.$$

From the free body situated on the left of a vertical cutting plane, passed anywhere between O and H and at a distance x from O, we find

$$T=T_1\cos\alpha, \quad \text{and} \quad M=\tfrac{3}{2}wax-\frac{wx^2}{2}-T_1x\sin\alpha,$$

whence $$\frac{dT}{dT_1}=\cos\alpha \quad \text{and} \quad \frac{dM}{dT_1}=-x\sin\alpha.$$

Also, from the free body extending from O to any vertical cutting plane passed between H and K, x measured from O as before, we have

$$T_2 = T, \quad \text{and} \quad M = \tfrac{3}{2}wax - \frac{wx^2}{2} - T_2 a \tan \alpha;$$

that is, $T_2 = T_1 \cos \alpha$, and $M = \tfrac{3}{2}wax - \dfrac{wx^2}{2} - T_1 a \sin \alpha$;

whence
$$\frac{dT_2}{dT_1} = \cos \alpha, \quad \text{and} \quad \frac{dM}{dT_1} = -a \sin \alpha.$$

These various functions of T_1 and corresponding derivatives being substituted in the detailed expression just given for $\dfrac{dU}{dT_1}$, the various integrations performed, and the result placed equal to zero; there is obtained, on solving for T_1,

$$T_1 = \frac{11wa^4 \sin \alpha}{6EI[2C_1 + (C_2 + C) \cos^2 \alpha] + 10a^3 \sin^2 \alpha}. \qquad (129)$$

If the temperature under load is different from the original temperature, the new value of T_1 may be obtained in the manner already illustrated in § 59.

61. Straight Prismatic Beam Supported by Two Piers and by Parabolic Cable. (Fig. 34.) The extremities of the beam rest on two fixed piers O and C, situated at the same level (and capable of preventing *upward* as well as downward move-

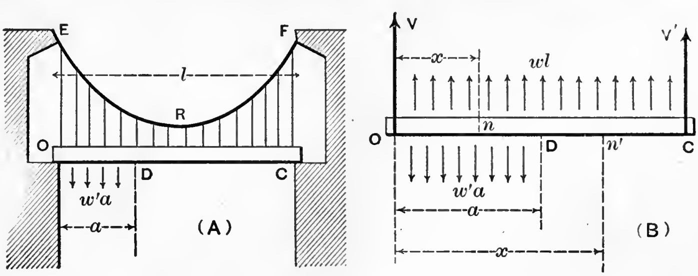

FIG. 34.

ment of the ends of the beam) and the beam is attached by (very numerous) vertical suspension rods, uniformly spaced along the horizontal, to a cable fastened at its extremities to the two fixed points E and F, which are at a common level

and vertically over O and C. It is stipulated that before any load acts on the beam (its own weight and that of rods and cable being neglected) all the suspension rods are barely taut, as also the cable (from one point of attachment of a rod to the next), and that the curve formed by the cable is a *common parabola,* with its *geometrical axis vertical* and passing midway between points E and F. R is the vertex of the curve. That is, the beam and the suspension rods and the cable are closely fitted to each other initially, and without strain, so that if a vertical load is gradually applied to the beam the stresses in the beam and rods and the parts of the cable increase progressively, and proportionally, from zero up to their final values. Evidently this presents a statically indeterminate problem; involving but one redundant element, if we consider the bending of the beam under load so slight that the curve of the cable remains practically a parabola. For this character of curve it is known from mechanics that the tensions in the suspension rods are all equal if these rods are very numerous and uniformly spaced along the horizontal, as in the present case. *The deformation* (stretching) *of the cable and suspension rods,* when the beam is loaded, *will here be neglected,* as having but small influence compared with the bending of the beam.

Let now a uniformly distributed load, W', of rate w' per foot run, be placed on the beam, extending from O along a portion OD, $=a$, of the length, l, of the beam; i.e., $W'=w'a$; w' and a being given. This will produce reactions, viz., V at O and V' at C, from the piers, and equal stresses in all the vertical rods at same rate w per horizontal foot. The system of forces now acting on the beam, as "free body," is shown at (B) in Fig. 34. There are *three* unknowns, viz., V, V', and w; while statics provides only *two* independent equations; hence the third must depend on elastic theory. As the single "redundant element" we may take either V, V', or w; let us choose w; then the others must be expressed as functions of w. From the free body at (B) (whole beam), by moments about C we find

$$V=\frac{w'a}{l}\left(l-\frac{a}{2}\right)-\frac{wl}{2}. \quad . \quad . \quad . \quad . \quad . \quad (130)$$

If the vertical rods are b ft. apart, the stress in some particular rod, which is to be taken as the redundant bar, is wb; and by the principle of least work (§ 29) we may put $\frac{dU}{d(wb)}=0$; i.e., $\frac{dU}{dw}=0$.

The internal work of shear being neglected, and also that of the cables and vertical rods (as previously implied), we have, therefore (E and I constant for beam),

$$\frac{dU}{dw}, =\frac{1}{EI}\left[\int_O^D M \cdot \frac{dM}{dw} \cdot dx+\int_D^C M \cdot \frac{dM}{dw} \cdot dx\right]=0. \quad (131)$$

For the dx at any point n on portion OD of beam, x being measured from O, taking On as a free body, we have [see also eq. (130)],

$$M=Vx+\frac{wx^2}{2}-\frac{w'x^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

$$=\left[\frac{w'a}{l}\left(l-\frac{a}{2}\right)-\frac{wl}{2}\right]x+\frac{wx^2}{2}-\frac{w'x^2}{2};$$

and hence
$$\frac{dM}{dw}=-\frac{l}{2}x+\frac{x^2}{2}, =\frac{x^2-lx}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

while for any point n' on portion DB, with x measured from O, and On' as free body,

$$M=Vx+\frac{wx^2}{2}-w'a\left[x-\frac{a}{2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

$$=\left[\frac{w'a}{l}\left(l-\frac{a}{2}\right)-\frac{wl}{2}\right]x+\frac{wx^2}{2}-w'a\left(x-\frac{a}{2}\right);$$

and
$$\therefore \frac{dM}{dw}=-\frac{l}{2}x+\frac{x^2}{2}, =\frac{x^2-lx}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

Hence substitution in eq. (131) gives

$$\int_0^a\left[w'ax-\frac{w'a^2x}{2l}-\frac{wlx}{2}+\frac{wx^2}{2}-\frac{w'x^2}{2}\right]\left(\frac{x^2-lx}{2}\right)dx$$

$$+\int_a^l\left[-\frac{w'a^2x}{2l}-\frac{wlx}{2}+\frac{wx^2}{2}+\frac{w'a^2}{2}\right]\left(\frac{x^2-lx}{2}\right)dx=0.$$

Performing the integrations between respective limits indicated, and reducing, we find

$$-2w'a^5+5w'a^4l-5w'a^2l^3+2wl^5=0;$$

solving which for w, writing m for the ratio $a \div l$, we have

$$w=\frac{m^2}{4}(4m^3-5m^2+10)w'. \quad . \quad . \quad . \quad . \quad (136)$$

With w known, V and V', and the moment and shear at any section of the beam, are easily computed, in a numerical case.

[**Note.** Evidently this problem involves the case of one variety of suspension bridge; in a more thorough treatment of which, however, the deformation of the cable should not be neglected; see Professor Hiroi's book on "Statically Indeterminate Structures" for a fuller investigation. The relation in eq. (136) is confirmed by a statement in Mr. H. M. Martin's small work of same title as Professor Hiroi's; and the results given in the remainder of this paragraph are quoted from Mr. Martin's book.]

If a **single concentrated load** P is carried on the beam in Fig. 34, instead of a distributed load, being placed at any distance c from the left support O, similar analysis (writing n for the ratio $c \div l$) leads to the result

$$w=5n(1-2n^2+n^3)\frac{P}{l}; \quad . \quad . \quad . \quad . \quad . \quad (137)$$

while the reaction V' at the support C is found to be

$$V'=\tfrac{1}{2}n(10n^2-5n^3-3)P. \quad . \quad . \quad . \quad . \quad (138)$$

For example, if $c=\frac{1}{2}l$, $V'=-\frac{9}{32}P$ (points downward).

If **the beam is built in horizontally** both at O and C ("fixed ends") instead of merely *resting* on the piers ("hinged ends"), with original close fit without strain as before, the same treatment may be adopted but the detail is much more cumbersome. We have now *three* redundant elements; viz., w and the moments M_o and M_c of the two internal couples in the beam at O and C. If any one of these, say M_o, be

regarded as a single redundant element, the stresses in the rods and the forces forming the other couple may be treated as *independent loads*, like P itself (see § 53), justifying us in placing $\frac{dU}{dM_o}=0$. Similarly, we have $\frac{dU}{dM_c}=0$, and $\frac{dU}{dw}=0$; and thus derive three independent equations, aside from the relations of statics. In this way w, M_o, and M_c can be found.

If in this case ("*fixed ends*") the load is uniformly distributed, viz., $W'=w'a$, extending a ft. from support O, we find (with $m=a\div l$),

$$w=m^3(10-15m+6m^2)w'; \quad . \quad . \quad . \quad . \quad (139)$$

and the shear in the vertical section of the beam close to support C is

$$J_c=m^3(7m-4-3m^2)w'l. \quad . \quad . \quad . \quad . \quad (140)$$

Also the value of M_c in section at C is

$$M_c=\tfrac{1}{2}m^3(1-2m+m^2)w'l^2. \quad . \quad . \quad . \quad . \quad (141)$$

(A positive result indicates tension in upper fibers.)

Again, if the beam in Fig. 34 (A) has **fixed ends** and carries a **single concentrated load,** P, at distance c from left end O by similar process we find (denoting $c\div l$ by n),

$$w=30n^2(1-2n+n^2)\frac{P}{l}, \quad . \quad . \quad . \quad . \quad (142)$$

$$J_c=n^2(28n-12-15n^2)P, \quad . \quad . \quad . \quad . \quad (143)$$

and

$$M_c=\tfrac{1}{2}n^2(3-8n+5n^2)Pl. \quad . \quad . \quad . \quad . \quad (144)$$

62. Double-Knee Beam; under Side Pressure. Constant E and I. Fig. 35 shows this "double-knee beam" as it may be called. It is a continuous beam, homogeneous, and of constant moment of inertia of section (I), consisting of three straight portions, continuous at the "knees," G and D; two vertical and one horizontal. Its own weight is to be neglected. Its extremities are hinged to two *immovable* piers, at same level, O and B; and before any load or force is brought upon it, it is under *no strain*; that is, it does not have to be "sprung" to fit it over the hinge pins. (It must be noted that there are no hinges at the corners G and D.)

Let now a horizontal wind pressure, wh, act against the left-hand side, of intensity of w lbs. per linear (vertical) foot. It is required to determine the horizontal component H, of the hinge reaction at O, the vertical component at O being V (components at B are H' and V') by the foregoing principles. Notation as in figure. Also let F = common sectional area of all parts of beam and let C_1 denote $\frac{h}{FE}$; and C_2 denote $\frac{l}{FE}$.

Denote by T_1, T_2, and T_3, the (unknown) thrusts in portions OG, GD, and DB, respectively; and neglect the internal work due to shears.

This is a statically indeterminate elastic structure; with only *one* redundant element, however, and this may be taken as either H, or H'; say, H.

We shall therefore put $\frac{dU}{dH}=0$; which may be justified in either of two ways. Note that of the five external forces, wh is fixed, i.e., given; and that V and V' are *also* fixed and are not affected by any imagined change in H or H'; since by taking moments about B we have $V = wh^2 \div 2l$. If the support at O consisted of a fixed horizontal smooth surface, a change in H would simply mean a shifting of joint O of beam along this surface and the beam would *still be supported;* H may therefore be considered as a load, independent of P, and must have a value corresponding to the fact that in the present case joint O of the beam *does not move horizontally* during the gradual application of the loads wh and H (H' would simply be a "*neutral*" reaction during this application; see § 21); i.e., $dU \div dH = 0$, by § 21.

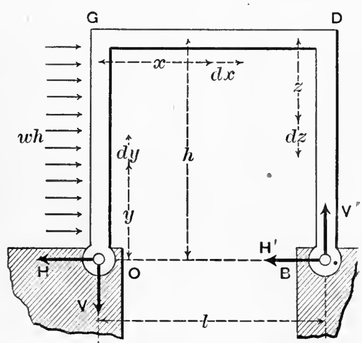

FIG. 35.

Again, from another point of view, consider that the beam is connected with the fixed support on the left, not by a hinge joint, but by *two inelastic bars,* one horizontal, the other vertical; then the stress in the former would be H; in the latter, V. These bars must be *inelastic,* otherwise joint O of the beam

would move during the gradual application of the load wh; and are now to be considered as part of the structure. Since the beam is only hinged to the support at B, and not "built-in," it is evident that the vertical bar is a *necessary* bar and not redundant, since without it the beam would not be supported. But the horizontal bar is a *redundant bar;* since, if it were omitted, the beam would still be supported, the joint O simply shifting to a position a little to the right of the original. Hence by § 29 we may put $dU \div dH = 0$; in which U should include the internal work of the two bars; but these being inelastic, such internal work is zero, so that in this case U simply refers to the internal work of the beam alone. We shall therefore use the relation $dU \div dH = 0$ and solve for H; remembering that V does not depend on H and hence need not be replaced by its value in terms of wh, until toward the close of the algebraic work.

From § 37, eq. (74), we have

$$U = \frac{C_1 T_1^2}{2} + \frac{C_2 T_2^2}{2} + \frac{C_1 T_3^2}{2} + \frac{1}{2EI}\int_O^B M^2 ds;$$

and hence

$$\frac{dU}{dH} = C_1 T_1 \frac{dT_1}{dH} + C_2 T_2 \frac{dT_2}{dH} + C_1 T_3 \frac{dT_3}{dH} + \frac{1}{EI}\int_O^B \left(M \cdot \frac{dM}{dH}\right) ds.$$

For the ds's situated along OG, $ds = dy$; $T_1 = V$; and $M = Hy - \frac{wy^2}{2}$. For those on GD, $ds = dx$; $T_2 = wh - H$; and $M = Hh - \frac{wh^2}{2} - Vx$; while along DB, $ds = dz$; $T_3 = V$; and $M = H(h-z) - wh\left(\frac{h}{2} - z\right) - Vl$.

We have, therefore,

$$\int_O^G M \frac{dM}{dH} ds = \int_0^h \left(Hy - \frac{wy^2}{2}\right)(+y)dy = \tfrac{1}{3}Hh^3 - \tfrac{1}{8}wh^4;$$

$$\int_G^D M \frac{dM}{dH} ds = \int_0^l \left(Hh - \frac{wh^2}{2} - Vx\right)(h)dx = Hh^2 l - \frac{wh^3 l}{2} - \frac{Vhl^2}{2};$$

$$\int_D^B M\frac{dM}{dH}ds=\int_{z=0}^{z=h}\left[H(h-z)-wh\left(\frac{h}{2}-z\right)-Vl\right](h-z)dz$$

$$=\int_0^h\left[H(h^2-2hz+z^2)-wh\left(\frac{h^2}{2}-\frac{3hz}{2}+z^2\right)-Vlh+Vlz\right]dz$$

$$=\frac{Hh^3}{3}-\frac{wh^4}{12}-\frac{Vlh^2}{2}.$$

Noting also that both $dT_1 \div dH$ and $dT_3 \div dH$ are zero, while $dT_2 \div dH = -1$, and substituting in the expression $dU \div dH = 0$, we find

$$H=\frac{\frac{11}{24}wh^4+\frac{3}{4}wh^3l+C_2EIwh}{C_2EI+h^2l+\frac{2}{3}h^3}. \quad . \quad . \quad . \quad . \quad (145)$$

A similar problem is treated by Mr. Mensch, on p. 80 of *Engineering News* of Feb., 1900. Besides the wind pressure on the vertical side Mr. Mensch also considers a uniformly distributed load along the whole length of the horizontal portion GD, of intensity of w_1 lbs. per linear foot. He not only neglects the internal work due to shears but also that due to the thrusts. In his problem the moment of inertia of the section of the portion GD is different, viz., I_1, from that, I, of the vertical portions of beam. His final result for H is, writing n for the ratio $I_1 \div I$,

$$H=\frac{\frac{11}{24}\,.\,n\,.\,wh^4+\frac{3}{4}wh^3l-\frac{1}{12}w_1hl^3}{h^2l+\frac{2}{3}\,.\,n\,.\,h^3}. \quad . \quad . \quad . \quad (146)$$

The results in eqs. (145) and (146) are seen to agree when in (145) we make $C_2=0$ (implying that the effect of the thrusts is negligible in this connection), and in (146) place w_1 equal to zero and I_1 equal to I.

If the horizontal side pressure is P lbs. *concentrated at* the corner G, instead of the distributed pressure wh, it may easily be proved that $H=H'=\frac{1}{2}P$.

Example. If, in the case of Fig. 35, the loading consists of one vertical load, P, applied vertically at the middle of GD, find the value of H; and also the vertical displacement of the point of application of P.

63. Loaded Davit of Uniform Section. Displacement of Extremity. The davit [see (A) in Fig. 36] consists of a straight vertical portion DK, built in vertically in the fixed support at D, and the curved portion KO, which is the quadrant of a circle of radius r. E and I are assumed constant throughout the whole length DKO. The tangent at K is vertical (before loading). Neglecting the weight of the davit itself, it is required to find the vertical displacement $\overline{Oa}$, $=y_1$, of the extremity O, as due to the gradual application of the vertical load P_1. Before loading, the extremity is at O; after loading, at O'.

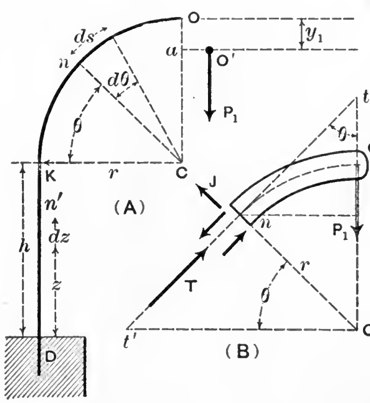

Fig. 36.

Since the projection of the displacement OO' upon the line of the external force or load P_1 is the quantity sought, viz., $\overline{Oa}$, $=y_1$, it may be determined as the derivative of the total internal work, U, of the beam, with respect to P_1; i.e., by eq. (25) of § 21, or $y_1=\frac{dU}{dP_1}$. Neglecting the work of shear, and also in this case the work of thrust, we have only the internal work due to bending moments, M; and hence may write [from eqs. (74) and (75), § 37],

$$y_1=\frac{dU}{dP_1}=\frac{1}{EI}\int_D^K M\cdot\frac{dM}{dP_1}\cdot ds+\frac{1}{EI}\int_K^O M\cdot\frac{dM}{dP_1}ds. \quad . \quad (147)$$

For the ds's situated along DK we measure z from D; so that $ds=dz$. For those along the curved part KO, using polar

co-ordinates, as shown, with angle θ measured from radius CK, we have $ds=r\,.\,d\theta$.

For any point n' on DK we have

$$M=P_1r;\quad \frac{dM}{dP_1}=r;\quad \text{and}\quad ds=dz;$$

while for any point n on the curve KO we have, from the free body nO, shown at (B) in Fig. 36,

$$M=P_1r\cos\theta;\quad \frac{dM}{dP_1}=r\cos\theta;\quad \text{and}\quad ds=rd\theta.$$

Substitution in eq. (147) gives (see integral forms in the Appendix),

$$y_1=\frac{P_1r^2}{EI}\int^h dz+\frac{P_1r^3}{EI}\int_0^{\pi/2}\cos^2\theta d\theta;$$

i.e.,
$$y_1=\frac{P_1r^2}{EI}\left[h+\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/2}\right]=\frac{P_1r^2}{EI}\left[h+\frac{\pi r}{4}\right]. \quad . \quad (147a)$$

To find the **horizontal** displacement of O, viz., $\overline{aO'}$ in (A), Fig. 36, since there is no external force at O pointing horizontally we have simply to introduce an imaginary force P_2 at O directed horizontally toward the right, and use the relation $\overline{aO'}=y_2=\frac{dU}{dP_2}$, in which finally P_2 is to be made equal to zero; thus following the procedure of §§ 23a and 40. With P_2 as well as P_1 in action we have

For any point n' on DK:

$$M=P_1r+P_2(r+h-z),\qquad \frac{dM}{dP_2}=r+h-z;\quad ds=dz.$$

For any point n on quadrant KO:

$$M=P_1r\cos\theta+P_2r[1-\sin\theta];\quad \frac{dM}{dP_2}=r(1-\sin\theta);\quad ds=rd\theta.$$

Substituting in $y_2=dM/dP_2$ (making, however, $P_2=0$ in each of the expressions for M, *before substitution*), we derive

$$y_2=\frac{P_1r}{EI}\left[\int_0^h(h+r-z)dz+r^2\int_0^{\pi/2}(\cos\theta-\sin\theta\cos\theta)d\theta\right]$$

$$=\frac{P_1r}{EI}\left[(r+h)h-\frac{h^2}{2}+r^2\left(\sin\theta-\frac{\sin^2\theta}{2}\right)\right]_0^{\pi/2}$$

$$=\frac{P_1r}{EI}\left[\left(rh+\frac{h^2}{2}\right)+r^2(1-\tfrac{1}{2})\right]=\frac{P_1r}{2EI}(h+r)^2. \quad . \quad . \quad . \quad (148)$$

64. Semicircular Curved Beam, or Arch Rib; Hinged at Ends. Single Eccentric Load. The curved beam is of semicircular form, radius $=r$, and is assumed to be homogeneous and of constant moment of inertia. That is, E and I are con-

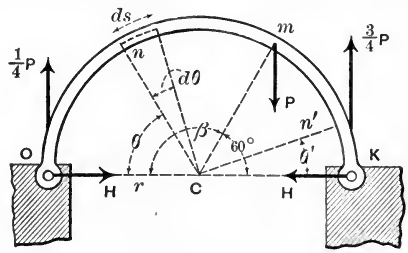

FIG. 37.

stant. Its own weight is neglected. It is hinged at its extremities to the fixed supports O and K, at the same level, being fitted to the hinges *without strain*, at temperature t_0, before loading. The temperature remaining constant, a single eccentric vertical load P is then placed at the point m, 60° from the right-hand horizontal radius, CK. Evidently the line of P is $\frac{3}{2}r$ from O and $\frac{1}{2}r$ from K. It is required to find the horizontal component, H, of the reaction at each hinge (evidently, in this case, H is the same for both).

Considering the whole rib as a free body, we easily find by ordinary statics (by moments about O and K) that the vertical components of the reactions at O and K are $\frac{1}{4}P$ and $\frac{3}{4}P$, respectively (they are independent of H); that is, H constitutes the only redundant element in the problem; and may be found by putting $\frac{dU}{dH}=0$ (bringing into play the same method of proof that was used in the fore part of § 62).

Here the work of the thrust, T, will be considered, as well as that of the "bending moment," M, but the work of shear will be neglected. With F as the sectional area of the beam, we have from eqs. (74) and 75),

$$\frac{dU}{dH}=\frac{1}{EI}\int_O^m M\frac{dM}{dH}ds+\frac{1}{EI}\int_K^m M\frac{dM}{dH}ds$$
$$+\frac{1}{FE}\int_O^m T\frac{dT}{dH}ds+\frac{1}{FE}\int_K^m T\frac{dT}{dH}ds.$$

For any point n between O and m, with θ as variable angle (see Fig. 37), we have, from the free body On (like that at (B) in Fig. 38),

$$M=\tfrac{1}{4}Pr(1-\cos\theta)-Hr\sin\theta;\quad \frac{dM}{dH}=-r\sin\theta;$$

$$T=\tfrac{1}{4}P\cos\theta+H\sin\theta;\quad \frac{dT}{dH}=\sin\theta;\quad ds=rd\theta;$$

$$J=\tfrac{1}{4}P\sin\theta-H\cos\theta;$$

while for any point n' on part Km, reckoning the corresponding variable angle θ' from radius CK, $n'K$ being the free body,

$$M=\tfrac{3}{4}Pr(1-\cos\theta')-Hr\sin\theta';\quad \frac{dM}{dH}=-r\sin\theta';$$

$$T=\tfrac{3}{4}P\cos\theta'+H\sin\theta';\quad \frac{dT}{dH}=\sin\theta';\quad ds=rd\theta';$$

$$J=\tfrac{3}{4}P\sin\theta'-H\cos\theta'.$$

Hence, with $ds=rd\theta$, or $rd\theta'$, we find

$$\frac{dU}{dH}=\frac{r^3}{EI}\int_0^{120°}(-\tfrac{1}{4}P\sin\theta+\tfrac{1}{4}P\sin\theta\cos\theta+H\sin^2\theta)d\theta$$
$$+\frac{r^3}{EI}\int_0^{60°}(-\tfrac{3}{4}P\sin\theta'+\tfrac{3}{4}P\sin\theta'\cos\theta'+H\sin^2\theta')d\theta'$$
$$+\frac{r}{FE}\int_0^{120°}(\tfrac{1}{4}P\sin\theta\cos\theta+H\sin^2\theta)d\theta$$
$$+\frac{r}{FE}\int_0^{60°}(\tfrac{3}{4}P\sin\theta'\cos\theta'+H\sin^2\theta')d\theta'.$$

Performing the integrations (see Appendix), collecting terms and reducing, and placing $dU/dH=0$, we find

$$\frac{r^3}{EI}\left(-0.3750P+\frac{\pi}{2}H\right)+\frac{r}{FE}\left(0.3750P+\frac{\pi}{2}H\right)=0. \quad . \quad (149)$$

Writing $I=Fk^2$, in which k is the radius of gyration of the cross-section of the beam, and solving for H there results

$$H=\left(\frac{r^2-k^2}{r^2+k^2}\right)(0.2386P). \quad . \quad . \quad . \quad . \quad (150)$$

H being found, the thrust, shear, and bending moment are easily determined for any section of beam.

The radial thickness of the beam being small compared with the radius r, the effect of considering the work of thrust in this semicircular form is extremely slight. Disregarding the work of thrust, which simply means dropping the term affected by the factor $(r\div FE)$ in eq. (149), or (which is the same thing) neglecting the term k^2 in (150) compared with r^2, we have

$$H=0.2386P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

instead of (150). For example, with $k=5$ in., and $r=20$ ft. $=$ 240 in., the two results differ by about one-tenth of one per cent, only. Even when the form of rib is a fairly flat circular arc, the effect of the thrust terms is relatively small.

The effects of a change of temperature will be treated in § 66.

65. Semicircular Arch Rib with Tie-Rod. In this case two fixed piers with smooth horizontal surfaces furnish *vertical* reactions, only, at the ends of the semicircular rib (constant E and I as before) whose extremities are connected by a horizontal bar or tie-rod. Piers at same level. (See (A), Fig. 38.)

When the load P, to be placed at the " crown," m, or highest point of the rib, is applied, the ends O and K of the rib spread apart slightly, and at the same time the horizontal tie-rod is stretched; the final stress in the latter being T_1, which is to be determined; it being understood that the rib and tie-rod are fitted closely together without strain, at some temperature t_0, before the load is applied, and that the temperature does

not change. Let F_1 denote the sectional area, l_1 the length, $=2r$, and E_1 the modulus of elasticity, of the tie-rod; C_1 representing $l_1 \div F_1E_1$.

Evidently, in this elastic system, the tie-rod is a redundant bar; there being but one redundant element in the case; and hence we write $\frac{dU}{dT_1}=0$, by § 29. The reactions (vertical) of the two smooth piers are each $P/2$, from symmetry, and independent of T_1. Considering the work of thrusts (including T_1)

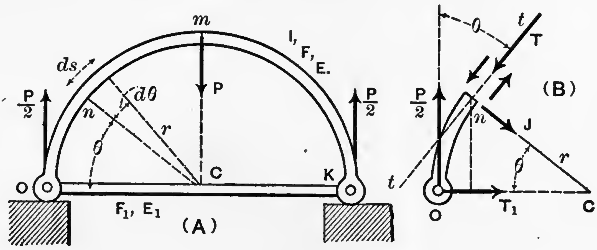

FIG. 38.

and of moments, but not that of shear, we have from eqs. (74) and (75), etc.,

$$\frac{dU}{dT_1}=\frac{d}{dT_1}\left(\frac{C_1T_1{}^2}{2}\right)+\frac{d}{dT_1}\left[\int_O^K\frac{M^2ds}{2EI}+\int_O^K\frac{T^2ds}{2FE}\right]=0$$

$$=C_1T_1+\frac{2}{EI}\int_O^m M\frac{dM}{dT_1}ds+\frac{2}{FE}\int_O^m T\frac{dT}{dT_1}ds=0. . \quad (152)$$

(On account of symmetry, we are enabled to put the internal work of the rib equal to twice that of the half between O and m.)

To find moments and thrusts in terms of T_1 consider the portion of the beam from O to any point n in the left-hand half, as a free body; see (B) in Fig. 38. M is found by taking moments about the gravity axis of the section at n; and T, the thrust, by summing components parallel to the tangent $t \ldots t$; thus excluding the shear J from either sum.

We have, then, for points on the quadrant Om,

$$M=\tfrac{1}{2}Pr(1-\cos\theta)-T_1r\sin\theta;\quad \frac{dM}{dT_1}=-r\sin\theta;$$

$$T=\tfrac{1}{2}P\cos\theta+T_1\sin\theta;\quad \frac{dT}{dT_1}=\sin\theta;\quad ds=rd\theta.$$

Substitution in eq. (152) gives

$$C_1T_1+\frac{2r^3}{EI}\int_0^{\pi/2}(-\tfrac{1}{2}P\sin\theta+\tfrac{1}{2}P\sin\theta\cos\theta+T_1\sin^2\theta)d\theta$$

$$+\frac{2r}{FE}\int_0^{\pi/2}(\tfrac{1}{2}P\sin\theta\cos\theta+T_1\sin^2\theta)d\theta=0. \quad . \quad (153)$$

Performing the integrations (see Appendix) and reducing,

$$C_1T_1-\frac{r^3}{EI}\cdot\frac{P}{2}+\frac{\pi r^3}{2}\frac{T_1}{EI}+\frac{rP}{2FE}+\frac{\pi rT_1}{2FE}=0,$$

i.e.,
$$T_1=\frac{\left[\frac{r^2}{EI}-\frac{1}{FE}\right]P}{\frac{4}{F_1E_1}+\frac{\pi r^2}{EI}+\frac{\pi}{FE}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

66. Temperature Stresses for Semicircular Arch Rib. If the rib is *hinged to two fixed piers,* as in Fig. 37, and the temperature t under load is higher than that t_0 of "erection" (when the rib was fitted in place without strain), we may consider the piers to furnish only the vertical reactions, and that the horizontal reactions are due to an *inelastic* horizontal rod or bar connecting O and K, whose coefficient of expansion is *zero,* and which was fitted in place without strain at the original lower temperature t_0. Then if the two parts, rib and bar, were to be fitted together at the higher temperature it would be found that the bar is *too short* by $\lambda_0=2r\eta(t-t_0)$, η being the coefficient of expansion of the material of the rib; since under no constraint the distance between the extremities of the rib has increased by that amount.

Hence by eq. (57) we may put $dU/dH=\lambda_0$, and in this, U must include the internal work of the bar; but as the latter is zero U comprises the internal work of the rib alone. Therefore, for the loading of § 64 and Fig. 37 we have only to put $2r\eta(t-t_0)$ in place of the zero in the right-hand member of eq. (149) and solve for H; obtaining (in case the work of both thrust and shear in the rib is disregarded),

$$H=0.2386P+\frac{2EIr\eta(t-t_0)}{r^3}. \quad . \quad . \quad . \quad (155)$$

If t_0 is lower than t, the same formula holds but $t-t_0$ is negative. If the rib in Fig. 37 carries no load, the H is entirely due to a change of temperature, if any; having a value

$$H=2(EI\div r^2)\eta(t-t_0).$$

In Fig. 38, where horizontal constraint is exerted upon the semicircular rib by an *actual* bar or tie-rod OK, which is elastic and has a coefficient of expansion η_1 (different, it may be, from that, η, of the material of the rib), it would be found that in fitting rib and rod together at a temperature t, higher than that, t_0, of erection, the rod is too short by an amount $\lambda_0=2r(\eta-\eta_1)(t-t_0)$ to connect the extremities of the rib. Hence we put $dU/dT_1=\lambda_0$, U including the internal work of the tie-rod. In other words, we write $2r(\eta-\eta_1)(t-t_0)$ in the place of the zero in the right-hand member of eq. (152); and obtain

$$T_1=\frac{\left[\frac{r^2}{EI}-\frac{1}{FE}\right]P+4(\eta-\eta_1)(t-t_0)}{\frac{4}{F_1E_1}+\frac{\pi r^2}{EI}+\frac{\pi}{FE}}. \quad . \quad . \quad (156)$$

In case the rib in Fig. 38 is *not loaded* this reduces to

$$T_1=[4(\eta-\eta_1)(t-t_0)]\div\left[\frac{4}{F_1E_1}+\frac{\pi r^2}{EI}+\frac{\pi}{FE}\right]. \quad . \quad (157)$$

For a *fall* of temperature $(t-t_0)$ would be negative, while if η_1 were numerically greater than η, $(\eta-\eta_1)$ would be negative.

67. Semicircular Curved Beam without Hinges; E and I Constant. Single Eccentric Load. [See (A) in Fig. 39.] This means that the semicircular arch rib is "*built in,*" or inserted rigidly in the two immovable piers (at same level) at O and K without strain before loading, and is continuous between. No change of temperature considered at present. Sectional area of rib is F. A vertical load P is to act on the rib at point m, 60° from the horizontal radius CK on the right. Hence, when the load is on, the cross-sectional plane of the rib at O (also at K) will be *the same as before loading,* and the center of gravity of the section at O (and at K) *will not move* as the load is applied.

At O, therefore, when the load is on, there will be in action a couple, of moment M_0; a horizontal shear, H; a vertical

thrust, V; all unknown in advance. Similarly, in the section at K, M_k, V', H'; also unknown. Since, for the whole beam as a free body, statics furnishes only three independent equations (one of which gives immediately $H'=H$), and there are six unknowns, it is evident that the beam is statically indeterminate to the extent of *three* redundant elements; and let V, H, and M_0 be chosen as these three redundant elements.

If the rib has been duly fitted in place without strain, and then the load P applied, it is evident that, the beam being *built in* at K, the pier at O might be entirely removed and the beam would still be supported, though becoming deformed (it would be a cantilever). The forces of the couple, also V and H, might then be gradually applied as *independent loads* with the

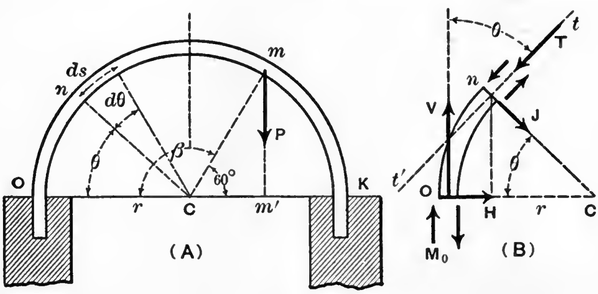

FIG. 39.

result that the sectional plane at O and the center of gravity of that section would finally be brought into the position where the pier O constrains them to be under actual conditions under load. In other words, we might remove pier O, let the rib remain unloaded at first, thus leaving the part O of beam in the same position as it will be finally under actual load, and then gradually and simultaneously apply all these "loads," viz., P, V, H, and the forces of the couple, until they reach their respective final values. During this gradual action the points of application of V and H *would not be displaced*, and the cross-sectional plane of the beam at O *would not turn*, thus justifying us in putting, separately [see eqs. (25) and (34)],

$$\frac{dU}{dM_0}=0; \quad \frac{dU}{dV}=0; \quad \text{and} \quad \frac{dU}{dH}=0. \quad . \quad . \quad . \quad (158)$$

It is to be noted that these are three independent equations. For instance, it is not necessary, before performing the differentiation implied in $dU/dH=0$, to express the other quantities M_0, and V, in terms of H; since the three quantities, M_0, V, and H, are independent of each other. For example, we might assume arbitrary values for M_0 and V, and then the imposing of the condition $dU/dH=0$ would simply give us such a value of H that during the gradual loading just described the point O of the beam would not be displaced horizontally; although it *would* be displaced vertically, and the section would turn through some angle ϕ; as dependent respectively on the arbitrary choice of V and M_0. Hence if there is to be neither vertical nor horizontal displacement at O, nor any turning of the plane of the cross-section, all three of the equations (158) must hold, independently. It remains to supply the detail of applying the three equations in (158), in order to determine the three unknown constants, M_0, V, and H.

Neglecting the internal work of shear, but considering that due to thrust T and moment M, we have

$$\frac{dU}{dM_0}=\frac{1}{EI}\int_O^K M\frac{dM}{dM_0}ds+\frac{1}{FE}\int_O^K T\frac{dT}{dM_0}ds=0; \quad . \quad (159)$$

$$\frac{dU}{dV}=\frac{1}{EI}\int_O^K M\frac{dM}{dV}ds+\frac{1}{FE}\int_O^K T\frac{dT}{dV}ds=0; \quad . \quad . \quad (160)$$

$$\frac{dU}{dH}=\frac{1}{EI}\int_O^K M\frac{dM}{dH}ds+\frac{1}{FE}\int_O^K T\frac{dT}{dH}ds=0. \quad . \quad . \quad (161)$$

Evidently the rib must be divided into two parts, $O\,..\,m$ and $m\,...\,K$, for the summations required. For any point n on $O\,...\,m$, taking the free body shown at (B) in Fig. 38, we have

$$M=M_0+Vr(1-\cos\theta)-Hr\sin\theta;$$

and hence

$$\frac{dM}{dM_0}=1; \quad \frac{dM}{dV}=r(1-\cos\theta); \quad \text{and} \quad \frac{dM}{dH}=-r\sin\theta.$$

Also $$T=V\cos\theta+H\sin\theta;$$

whence $$\frac{dT}{dM_0}=0; \quad \frac{dT}{dV}=\cos\theta; \quad \text{and} \quad \frac{dT}{dH}=\sin\theta.$$

For any point n' between m and K, with $O \ldots n'$ as free body, still reckoning angle θ from radius CO (noting that the distance of P from O is $\frac{3}{2}r$; from K, $\frac{1}{2}r$), we have

$$M = M_0 + Vr(1-\cos\theta) - Hr\sin\theta + Pr(\tfrac{1}{2}+\cos\theta);$$

hence

$$\frac{dM}{dM_0}=1;\quad \frac{dM}{dV}=r(1-\cos\theta);\quad \frac{dM}{dH}=-r\sin\theta;$$

$$T = V\cos\theta + H\sin\theta - P\cos\theta;$$

and therefore

$$\frac{dT}{dM_0}=0;\quad \frac{dT}{dV}=\cos\theta;\quad \text{and}\quad \frac{dT}{dH}=\sin\theta.$$

First substituting in eq. (159), noting that the second integral disappears since $dT/dM_0=0$, we have, with $ds=rd\theta$,

$$\frac{r}{EI}\int_{0°}^{120°}(M_0+Vr-Vr\cos\theta-Hr\sin\theta)d\theta$$

$$+\frac{r}{EI}\int_{120°}^{180°}(M_0+Vr-Vr\cos\theta-Hr\sin\theta+\tfrac{1}{2}P+Pr\cos\theta)d\theta=0.$$

Performing the integrations (see Appendix), noting that $\sin 120° = 0.8660$, $\cos 120° = -0.500$, arc $120° = \frac{2}{3}\pi$, and arc $180° = \pi$, we obtain after final reduction,

$$\pi M_0 + \pi r V - 2rH + \frac{\pi r}{6}P - 0.8660rP = 0; \quad . \quad . \quad (162)$$

which is the first of the necessary three simultaneous equations involving the three unknowns M_0, H, and V.

Next we have from eq. (160),

$$\frac{1}{EI}\int_{0°}^{120°}\left[M\frac{dM}{dV}+T\frac{dT}{dV}\right]ds+\frac{1}{FE}\int_{120°}^{180°}\left[M\frac{dM}{dV}+T\frac{dT}{dV}\right]ds=0.$$

Substitution in this equation of the values for M, dM/dV, dT/dV already obtained, having regard to the portion of beam within which each applies, with $ds=rd\theta$, leads, after integration and reduction, to the result,

$$\frac{\pi r}{I}M_0+\frac{\pi}{2}\left[\frac{3r^2}{I}+\frac{1}{F}\right]V-\frac{2r^2}{I}H-\left[\frac{0.6495r^2}{I}+\frac{0.7401}{F}\right]P=0. \quad (163)$$

Similarly, the result of substituting in eq. (161), or $dU/dH = 0$, i.e., in

$$\frac{1}{EI}\int_{0^\circ}^{120^\circ}\left[M\frac{dM}{dH}+T\frac{dT}{dH}\right]ds+\frac{1}{FE}\int_{120^\circ}^{180^\circ}\left[M\frac{dM}{dH}+T\frac{dT}{dH}\right]ds=0;$$

after integration and reduction is found to be

$$-\frac{2r}{I}M_0-\frac{2r^2}{I}V+\left[\frac{\pi r^2}{2I}+\frac{\pi}{2F}\right]H+\left[\frac{r^2}{8I}+\frac{3}{8F}\right]P=0. \qquad (164)$$

The three simultaneous eqs. (162), (163), and (164), can now be solved for M_0, V, and H.

Work of Thrust Neglected. With such a large angle as 180° for a circular arch rib the effect of considering the work of thrust is extremely small; so the terms arising from that source will now be neglected (that is, the terms involving the divisor F). This causes I to disappear from all three equations. By elimination between (162) and (163) we find

$$V=0.1955P \text{ (lbs.)}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (165)$$

and further combination gives

$$H=0.312P \text{ (lbs.)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (166)$$

and

$$M_0=0.1127Pr \text{ (in.-lbs.)}. \quad . \quad . \quad . \quad . \quad . \qquad (167)$$

Work of Thrust Retained. If the terms containing the divisor F be retained and eqs. (162), (163), and (164) be combined without modification (that is, the work of thrust is now considered; though not that of shear) the result of elimination will be

$$V=\frac{\left(0.3071+\frac{0.7401I}{Fr^2}\right)P}{\frac{\pi}{2}\quad 1-\frac{I}{Fr^2}}; \quad . \quad , \quad . \quad . \quad . \qquad (168)$$

and

$$H=\frac{\left(0.0466-\frac{3I}{Fr^2}\right)P}{0.1491+\frac{\pi}{4}\cdot\frac{I}{Fr^2}}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (169)$$

and, further, regarding both V and H as having been found,

$$M_0=\left[\frac{0.3424}{\pi}\cdot P+\frac{2}{\pi}H-V\right]r. \quad . \quad . \quad . \qquad (170)$$

If in eqs. (168) and (169) we write $I=Fk^2$, where k is the "radius of gyration" of the cross-section of rib, the fraction $I\div Fr^2$ reduces to k^2/r^2, which ratio is extremely small; and the terms containing it are so small, relatively (in this case of a *semicircular* rib with radial thickness small compared with the radius r of the semicircle) that the omission of the terms involving this fraction, $I\div Fr^2$, in the equations mentioned, leads to no practical error and gives rise to the results already obtained in eqs. (165), (166), and (167). For results in case the rib is segmental instead of semicircular, and for temperature stresses, see §§ 68 and 69.

68. Segmental Circular Arch Rib Hinged at Ends. Constant E and I. Fig. 40 shows this case, with a single concentrated load P at any point m, the radius to which, Cm, makes an angle β with the horizontal radius CO on the left (β may be $<$ or $>90°$). As before, the rib is homogeneous (constant E)

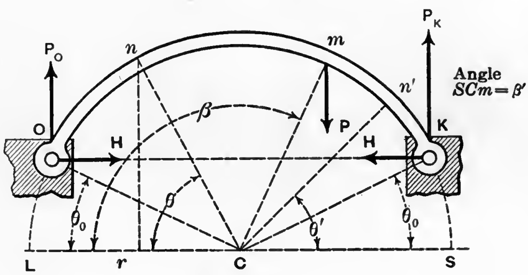

FIG. 40.

and its cross-section has a constant I; the radial thickness being small compared with the span OK. The radius, Cn of any point n of the axis of rib on the left of m is at some angle θ with the horizontal measured from CL; while for any point n' on the right the corresponding angle is called θ', measured from CS. The angle SCm will be denoted by β'; i.e., $\beta'=\pi-\beta$. The axis of the rib is the arc of a circle of radius r, and is symmetrical about the vertical line through center C. With the one vertical load, P, as shown, the reaction of each hinge

is oblique (since the rib tends to spread), and will be replaced by its vertical and horizontal components. From Σ (hor. compon.) $=0$ the two horizontal components are equal to each other (value H). The vertical reactions P_0 and P_K are easily found by statics. With the whole rib as free body, moments about hinge K gives, after canceling the r,

$$P_0 = \frac{(\cos\theta_0 + \cos\beta)P}{2\cos\theta_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

The piers are immovable and it is assumed that the rib has been fitted in place originally without strain and the load applied later; and that no change in temperature has occurred. The value of H is the statically redundant element (there is but one such element in this problem) and may be found, as already explained in § 62, by putting $dU/dH=0$. As before, let ds, $=rd\theta$ (or $rd\theta'$, on right of P), be an elementary portion of the axis of the rib, at any point n (or n'); then, considering the internal work of bending, and of thrust, but not that due to shear, we have [see eq. (75)],

$$\frac{dU}{dH} = \frac{1}{EI}\int_O^K M\frac{dM}{dH}ds + \frac{1}{FE}\int_O^K T\frac{dT}{dH}ds = 0, \quad . \quad (172)$$

the summation being extended over the whole length of the axis of rib, from O to K. Detail as follows:

For any ds occurring between O and m we have, considering portion On as a free body,

$$M = P_0 r(\cos\theta_0 - \cos\theta) - Hr(\sin\theta - \sin\theta_0), \quad . \quad (173)$$

and
$$\frac{dM}{dH} = -r(\sin\theta - \sin\theta_0), \quad . \quad . \quad . \quad . \quad (173a)$$

also
$$T = P_0 \cos\theta + H\sin\theta, \quad . \quad . \quad . \quad . \quad (174)$$

and
$$\frac{dT}{dH} = +\sin\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174a)$$

and the limits for θ are from $\theta=\theta_0$ to $\theta=\beta$.

Similar expressions for M and T will hold for any ds between m and K, if for P_0, θ, and $d\theta$, we write P_k, θ', and $d\theta'$, respectively; and the limits for θ' will be θ_0 and β'.

Substitution being made in eq. (172) we obtain

$$-\frac{P_0 r^3}{EI}\int_{\theta_0}^{\beta}[\cos\theta_0\sin\theta-\sin\theta\cos\theta-\sin\theta_0\cos\theta_0+\sin\theta_0\cos\theta]d\theta$$

$$+\frac{Hr^3}{EI}\int_{\theta_0}^{\beta}[\sin^2\theta-2\sin\theta_0\sin\theta+\sin^2\theta_0]d\theta$$

$$+\frac{P_0 r}{FE}\int_{\theta_0}^{\beta}\sin\theta\cos\theta\,d\theta+\frac{Hr}{FE}\int_{\theta_0}^{\beta}\sin^2\theta d\theta$$

$$-\frac{P_K r^3}{EI}\int_{\theta_0}^{\beta'}[\cos\theta_0\sin\theta'-\sin\theta'\cos\theta'-\sin\theta_0\cos\theta_0+\sin\theta_0\cos\theta']d\theta'$$

$$+\frac{Hr^3}{EI}\int_{\theta_0}^{\beta'}[\sin^2\theta'-2\sin\theta_0\sin\theta'+\sin^2\theta_0]d\theta'$$

$$+\frac{P_K r}{FE}\int_{\theta_0}^{\beta'}\sin\theta'\cos\theta' d\theta'+\frac{Hr}{FE}\int_{\theta_0}^{\beta_{\prime}}\sin^2\theta'\,d\theta'=0. \quad . \quad . \quad . \qquad (175)$$

To effect a solution of eq. (175) for H, we first perform the integrations indicated; which is fairly simple, since each integral contains but one variable, θ or θ', and the forms may be found in the table of integrals in the Appendix. For P_K, $P-P_0$ is now substituted; and then, for P_0 (in each term containing it), the value given in eq. (171). Terms are now collected; and use made, for further reduction, of the following relations: $\beta'=\pi-\beta$; $\sin\beta'=\sin\beta$; $\cos\beta'=-\cos\beta$; $\sin 2\beta'=-\sin 2\beta$ and $I=Fk^2$.

If now we denote by A the expression

$$\tfrac{1}{2}(\sin^2\beta-3\sin^2\theta_0)+\sin\theta_0[\sin\beta-\theta_1\cos\theta_0+\beta_1\cos\beta],$$

in which $\theta_1=\frac{\pi}{2}-\theta_0$ and $\beta_1=\frac{\pi}{2}-\beta$, final solution leads to the result,

$$H=\frac{\left(A-\frac{1}{2}\cdot\frac{k^2}{r^2}[\sin^2\beta-\sin^2\theta_0]\right)P}{\theta_1(1+2\sin^2\theta_0)-\frac{3}{2}\sin 2\theta_0+\frac{k^2}{r^2}(\theta_0+\sin 2\theta_0)}. \quad . \quad . \qquad \textbf{(176)}$$

With H known, the values of moment, thrust, and shear are easily found for any section of the rib.

If the internal work of thrust is disregarded, the result for H is that due to the omission of the terms in (176) containing the factor k^2/r^2; i.e.,

$$H=\frac{AP}{\theta_1(1+2\sin^2\theta_0)-\frac{3}{2}\sin 2\theta_0}. \quad . \quad . \quad . \quad (177)$$

As the arc of the arch rib approaches a semicircle the difference between (176) and (177) becomes very slight.

69. Temperature Stresses in Circular Segmental Arch Rib of Two Hinges. With load P still in place (in Fig. 40), the length of span being $\overline{OK}=2r\cos\theta_0$, if the temperature rises to a value t higher than that, t_0, of erection, the piers being unyielding, we may prove, as in the first part of § 66 that the value of H is obtained by putting dU/dH equal to the amount λ_0, viz., $\eta(2r\cos\theta_0)(t-t_0)$, that the distance between the hinges O and K, of the rib, would have increased, had the rib been free to expand with rising temperature (instead of putting $dU/dH=0$). Filling in details in the same manner as in § 68 we obtain (for meaning of A, see end of § 68),

$$H=\frac{\left(A-\frac{1}{2}\frac{k^2}{r^2}[\sin^2\beta-\sin^2\theta_0]\right)P+\frac{FEk^2}{r^2}(2\eta\cos\theta_0)(t-t_0)}{\theta_1(1+2\sin^2\theta_0)-\frac{3}{2}\sin 2\theta_0+\frac{k^2}{r^2}(\theta_1+\sin 2\theta_0)}; \quad (178)$$

in which if P be made zero the value of H is that resulting from the change of temperature alone. Of course in that case P_0, P_K, and P (in Fig. 40) would drop out; and the moment, thrust, and shear at any section, due to the H alone, would be the elements from which the fiber stresses, which now may be called " temperature stresses," would be computed.

For a *fall* of temperature below that, t_0, of erection, the quantity $(t-t_0)$ becomes negative in eq. (178).

70. Segmental Circular Arch Rib without Hinges. E and I Constant. Ends Fixed in Immovable Piers. Here, as in § 67, it is stipulated that the ends of the rib are fixed, or " built in " (or " encastré ") in the piers or supports *without strain* before any loading; that is, the rib is put in place originally without initial stresses. Fig. 41 shows this case, a symmetrical arrange-

ment with extremities O and K at the same level; E and I constant. A single vertical load P is placed at any point m, the radius to which, Cm, makes an angle β with the left-hand horizontal radius LC. Each of the extreme radii CO and CK makes an angle θ_0 with the horizontal; and the angle LCK (which $=\pi-\theta_0$) will be called μ.

The variable angle OCn for any point n, or element ds, of the axis of rib on the left of m will be called θ. For any point n', *on the right of* m, *also*, the angle θ will be measured from LC in the present analysis.

When the load P is in place (weight of rib itself neglected) stresses will be induced in the cross-section at O equivalent to

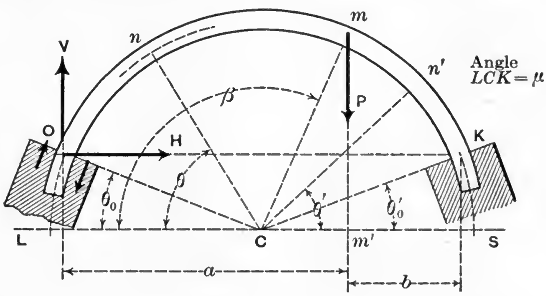

FIG. 41.

a couple of moment M_0, a shear, and a thrust. But the shear and thrust may be replaced by a single resultant passing through the center of gravity of the section at O, and this resultant may then be decomposed into a horizontal component H and a vertical component V. These are shown in Fig. 41 as acting on that end of the rib. Similarly, in the cross-section at K, the stresses acting are equivalent to a couple of moment M_K, a horizontal force H_K and a vertical force V_K; these latter acting through the center of gravity of the cross-section at K. These two couples, the forces H, H_K, V, V_K, and the load P, form a system of forces holding the whole rib in equilibrium when considered as a free body. It is seen that (P being given) six unknowns are involved, viz., M_0, M_K, H, H_K, V, and V_K. Since statics furnishes but three independent equations, the remaining three needed for complete solution must be based on the theory of elasticity. As in § 67 we note that there

are here three redundant elements involved, which could be any three of the above six quantities. Our present treatment will select M_0, H, and V, as the three redundant elements; and the analysis will be so shaped as to exclude the quantities M_K, H_K, and V_K from the equations brought into play for determining M_0, H, and V. We shall, therefore (see § 67), write the three relations

$$\frac{dU}{dM_0}=0; \quad \frac{dU}{dV}=0; \quad \text{and} \quad \frac{dU}{dH}=0; \quad . \quad . \quad . \quad (179)$$

and from them determine M_0, V, and H, in terms of P.

Details as follows: From the free body $O \ldots n$, we have for the moment and thrust at any section n between O and m,

$$M=M_0+Vr(\cos\theta_0-\cos\theta)-Hr(\sin\theta-\sin\theta_0); \quad . \quad (180)$$

$$T=V\cos\theta+H\sin\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

whence

$$\frac{dM}{dM_0}=1; \quad \frac{dM}{dV}=r(\cos\theta_0-\cos\theta); \quad \frac{dM}{dH}=-r(\sin\theta-\sin\theta_0);$$

and

$$\frac{dT}{dM_0}=0; \quad \frac{dT}{dV}=\cos\theta; \quad \frac{dT}{dH}=\sin\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

whereas, for any section n' between m and K we have, from $O \ldots n'$ as a free body,

$$M=M_0+Vr(\cos\theta_0-\cos\theta)-Hr(\sin\theta-\sin\theta_0)-Pr(\cos\beta-\cos\theta);$$

$$T=H\sin\theta+V\cos\theta-P\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

whence

$$\frac{dM}{dM_0}=1; \quad \frac{dM}{dV}=r(\cos\theta_0-\cos\theta); \quad \frac{dM}{dH}=-r(\sin\theta-\sin\theta_0);$$

and

$$\frac{dT}{dM_0}=0; \quad \frac{dT}{dV}=\cos\theta; \quad \frac{dT}{dH}=\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

First Equation. Expanding the first statement of eq. (179), neglecting the work of shear, and indicating limits of itegration by limiting values of the variable angle θ for the two portions Om and mK of the rib, we have

$$\frac{dU}{dM_0}=\frac{1}{EI}\int_{\theta_0}^{\beta}M\frac{dM}{dM_0}\cdot ds+\frac{1}{FE}\int_{\theta_0}^{\beta}T\frac{dT}{dM_0}ds$$
$$+\frac{1}{EI}\int_{\beta}^{\mu}M\frac{dM}{dM_0}ds+\frac{1}{FE}\int_{\beta}^{\mu}T\frac{dT}{dM_0}ds=0. \quad . \quad (185)$$

But, from eqs. (182) and (184), dT/dM_0 is zero both on $O \ldots m$ and on $m \ldots K$: hence the second and fourth integrals on the right drop out. Substituting, now, the proper values of M and dM/dM_0, from eqs. (180) to (184), in the first and second integrals, putting $ds=r \, . \, d\theta$, and denoting the expression forming the right-hand member of eq. (180) by Z, we have, after canceling the r/EI,

$$\int_{\theta_0}^{\beta}Zd\theta+\int_{\beta}^{\mu}[Z-Pr(\cos\beta-\cos\theta)]d\theta=0. \quad . \quad (186)$$

Note. Now $\int_{\theta_0}^{\beta}Zd\theta+\int_{\beta}^{\mu}Zd\theta$ is the same thing as $\int_{\theta_0}^{\mu}Zd\theta$; whence we have

$$\int_{\theta_0}^{\mu}[M_0+Vr(\cos\theta_0-\cos\theta)-Hr(\sin\theta-\sin\theta_0)]d\theta$$
$$-Pr\int_{\beta}^{\mu}(\cos\beta-\cos\theta)d\theta=0. \quad . \quad (187)$$

This equation contains but one variable, θ. After integration, with insertion of limits, further reduction is brought about by the use of the relations: $\mu=\pi-\theta_0$; $\sin\mu=\sin\theta_0$; $\cos\mu=-\cos\theta_0$; we shall also write $\frac{\pi}{2}-\theta_0=\theta_1$, and $\frac{\pi}{2}-\beta=\beta_1$. The final form then becomes

$$2\theta_1(M_0+Vr\cos\theta_0)-2Hr[\cos\theta_0-\theta_1\sin\theta_0]$$
$$-Pr[(\theta_1+\beta_1)\cos\beta+\sin\beta-\sin\theta_0)=0; \quad . \quad \mathbf{(188)}$$

which is the *first* of the required three equations containing the three *unknown constants* M_0, V, and H (and no others). The *given* quantities are P, θ_0, θ_1, β, β_1, E, I, F, and r.

Second Equation. Expanding $dU/dV=0$ [of eqs. (179)] we have (neglecting work of thrust)

$$\frac{dU}{dV}=\frac{1}{EI}\int_{\theta_0}^{\beta}M\frac{dM}{dV}ds+\frac{1}{FE}\int_{\theta_0}^{\beta}T\frac{dT}{dV}ds$$
$$+\frac{1}{EI}\int_{\beta}^{\mu}M\frac{dM}{dV}ds+\frac{1}{FE}\int_{\beta}^{\mu}T\frac{dT}{dV}ds=0. \quad . \quad (189)$$

Values for M, T, dM/dV, and dT/dV, from eqs. (180) to (184), are now inserted in their proper places in (189); and advantage is taken of the relation pointed out in the "*Note*" following eq. (186). Whence, with $ds=rd\theta$, we obtain

$$\frac{r^2}{EI}\int_{\theta_0}^{\mu}[M_0+Vr(\cos\theta_0-\cos\theta)-Hr(\sin\theta-\sin\theta_0)](\cos\theta_0-\cos\theta)d\theta$$
$$-\frac{r^3}{EI}\int_{\beta}^{\mu}P(\cos\beta-\cos\theta)(\cos\theta_0-\cos\theta)d\theta$$
$$+\frac{r}{FE}\left[\int_{\theta_0}^{\mu}(H\sin\theta+V\cos\theta)\cos\theta d\theta-\int_{\beta}^{\mu}P\cos^2\theta d\theta\right]=0. \quad (190)$$

Integration between the limits shown, and the aid of relations already mentioned, reduce this to the form (Fk^2 having replaced I),

$$2\theta_1M_0\cos\theta_0-2Hr(\cos^2\theta_0-\theta_1\sin\theta_0\cos\theta_0)$$
$$+Vr\left[2\theta_1\cos^2\theta_0+\left(1+\frac{k^2}{r^2}\right)(\theta_1-\tfrac{1}{2}\sin 2\theta_0)\right]$$
$$-Pr\Big([(\theta_1+\beta_1)\cos\beta-\sin\theta_0+\sin\beta]\cos\theta_0-(\sin\theta_0-\sin\beta)\cos\beta\Big)$$
$$-\frac{Pr}{2}\left(1+\frac{k^2}{r^2}\right)[\theta_1+\beta_1-\tfrac{1}{2}(\sin 2\theta_0+\sin 2\beta)]=0; \quad . \quad . \quad . \quad . \quad (191)$$

which is the *second* equation needed; between M_0, V, and H.

Third Equation. Finally, the detail of $dU/dH=0$ [see eqs. (179)] is as follows (work of shear omitted):

$$\frac{dU}{dH}=\frac{1}{EI}\int_{\theta_0}^{\beta}M\cdot\frac{dM}{dH}ds+\frac{1}{EI}\int_{\beta}^{\mu}M\cdot\frac{dM}{dH}ds$$
$$+\frac{1}{FE}\int_{\theta_0}^{\beta}T\cdot\frac{dT}{dH}ds+\frac{1}{FE}\int_{\beta}^{\mu}T\frac{dT}{dH}ds=0. \quad . \quad (192)$$

Substituting values from eqs. (180) to (184) and utilizing the relation of the "*Note*" following eq. (186), with $ds = r \, . \, d\theta$, we have

$$\frac{r^2}{EI}\int_{\theta_0}^{\mu}[M_0 + Vr(\cos\theta_0 - \cos\theta) - Hr(\sin\theta - \sin\theta_0)](\sin\theta_0 - \sin\theta)d\theta$$

$$-\frac{r^3}{EI}\int_{\beta}^{\mu}P(\cos\beta - \cos\theta)(\sin\theta_0 - \sin\theta)d\theta$$

$$+\frac{r}{FE}\left[\int_{\theta_0}^{\mu}(H\sin\theta + V\cos\theta)\sin\theta d\theta - \int_{\beta}^{\mu}P\cos\theta\sin\theta d\theta\right] = 0. \quad (193)$$

After integration and reduction this becomes *

$$2M_0(\theta_0\sin\theta_0 - \cos\theta_0) + 2Vr\cos\theta_0(\theta_1\sin\theta_0 - \cos\theta_0)$$

$$+Hr\left[\left(1+\frac{k^2}{r^2}\right)(\theta_1 + \tfrac{1}{2}\sin 2\theta_0) - 2\sin 2\theta_0 + 2\theta_1\sin^2\theta_0\right]$$

$$-Pr[(\theta_1+\beta_1)\sin\theta_0\cos\beta + \sin\theta_0\sin\beta - \cos\beta\,(\cos\theta_0 + \cos\beta)]$$

$$-\frac{Pr}{2}\left[\left(1+\frac{k^2}{r^2}\right)(\sin^2\theta_0 - \sin^2\beta) - 2\sin^2\theta_0\right] = 0. \quad . \quad . \quad . \quad \mathbf{(194)}$$

This is the *third* equation needed, of the three between the unknown constants M_0, V, and H.

Elimination. By a fortunate coincidence eqs. (188) and (191) happen to be so constituted that both M_0 and H may be eliminated by a single operation, viz.: Multiply eq. (188) throughout by $-\cos\theta_0$ and add the resulting equation to (191), member to member. The equation thus produced contains neither M_0 nor H; and solved for V gives rise to

$$V = \frac{\left[\cos\beta\,(\sin\beta - \sin\theta_0) + \left(1+\frac{k^2}{r^2}\right)B\right]P}{\left(1+\frac{k^2}{r^2}\right)(\theta_1 - \frac{1}{2}\sin 2\theta_0)}. \quad . \quad \mathbf{(195)}$$

in which the symbol B stands for the expression *

$$\tfrac{1}{2}(\theta_1 + \beta_1 - \tfrac{1}{2}\sin 2\theta_0 - \tfrac{1}{2}\sin 2\beta). \quad . \quad . \quad . \quad (196)$$

Similarly, M_0 and V may be eliminated from the two equations (188) and (194) by one operation, viz.: Solve each of the

* For meaning of θ_1 and β_1 see after eq. (187).

equations (188) and (194) for M_0 and equate the expressions so derived. The resulting equation contains H, but not M_0 nor V. Solving for H, we obtain, using the symbol C to denote the expression

$$\cos\theta_0(\beta_1\cos\beta+\sin\beta)-\theta_1\cos^2\beta-\tfrac{1}{2}\sin 2\theta_0),$$

$$H=\frac{\left[C-\left(1+\frac{k^2}{r^2}\right)(\sin^2\beta-\sin^2\theta_0)\frac{\theta_1}{2}\right]P}{\left(1+\frac{k^2}{r^2}\right)(\theta_1+\frac{1}{2}\sin 2\theta_0)\theta_1-2\cos^2\theta_0} \quad . \quad . \quad . \quad \textbf{(197)}$$

Since H and V are now known in terms of P a solution of eq. (188) for M_0 will give the latter quantity in known terms; thus

$$M_0=\left[\left(\frac{\cos\theta_0}{\theta_1}-\sin\theta_0\right)H-(\cos\theta_0)V\right.$$
$$\left.+\left([\theta_1+\beta_1]\cos\beta+\sin\beta-\sin\theta_0\right)\frac{P}{2\theta_1}\right]r. \quad . \quad (198)$$

Great complication would result from the insertion in (188) of the *algebraic* values of H and V from eqs. (195) and (197), and no useful purpose would be served thereby. When H and V have been determined *numerically* in any given case, placing their values in (198) leads directly to the numerical value of M_0.

With M_0, V, and H known, the moment, thrust, and shear (M, T, and J) may be found at any cross-section of the rib (including that at K, where we find M_K, V_K, and H_K) by the conditions of equilibrium of a free body extending from O to the section in question (evidently $H_K=H$). Hence all stresses are now determinate, as to a single vertical load P applied at any point m of the rib, Fig. 41. It is understood that the temperature when P is in place is the same as that, t_0, of erection.

If in the foregoing expressions for M_0, V, and H, the ratio $\frac{k^2}{r^2}$, wherever occurring, is placed equal to zero, the results are those that would be obtained if, besides neglecting the work of shear, we had also neglected the work of thrust. As the form of the rib approaches a semicircle, the work of thrust is of less and less consequence.

71. Temperature Stresses in Segmental Circular Arch Rib of No Hinges. Fixed Supports. E and I Constant. (See Fig. 41.) With no load whatever on the rib, t_0 being the temperature of placing in position ("erection"), there being then no constraint, that is, no "initial strains," stresses are induced if the temperature rises to some higher figure t; due to attempted expansion on the part of the rib. At each of the sections O and K, a couple, a thrust, and a shear, are developed. The thrust and shear being combined into a resultant, and that resultant decomposed along the vertical and horizontal, evidently, from the equilibrium of the whole rib (from the conditions of statics) each vertical component is zero and the two horizontal components are equal $(=H)$ while the moments of the two couples are equal $(=M_0)$. There are, therefore, two redundant elements, M_0 and H.

If η is the coefficient of expansion of the material of the rib, the point O of the rib would have moved horizontally toward the left a distance, λ_0, equal to $\eta(t-t_0)\,2r\cos\theta_0$ (see Fig. 41) if the rib had been free to change form by the removal of the left-hand pier (the other pier remaining fixed). Also, by the fixity of the left-hand pier, the *turning* of the plane of the cross-section of rib at O, which would have occurred if there had been a fixed *hinge* joint at O, is prevented. Hence by reasoning similar to that on p. 102 (see also § 66) we must have the conditions:*

$$\frac{dU}{dM_0}=0; \quad \text{and} \quad \frac{dU}{dH}=\lambda_0, \quad . \quad . \quad . \quad . \quad . \quad (199)$$

which will serve to determine the two unknowns, M_0 and H.

Detail as follows: From the free body $O \dots n$, n being any point of axis of rib between O and K, we find

$$M=M_0-Hr(\sin\theta-\sin\theta_0) \quad . \quad . \quad . \quad . \quad (200)$$

and

$$T=H\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

$$\therefore \frac{dM}{dM_0}=1; \quad \frac{dM}{dH}=-r(\sin\theta-\sin\theta_0); \quad . \quad . \quad (202)$$

$$\frac{dT}{dM_0}=0; \quad \text{and} \quad \frac{dT}{dH}=\sin\theta. \quad . \quad . \quad . \quad . \quad (203)$$

* See Note D in Appendix.

In making each of the summations called for in eqs. (199) advantage can be taken of the symmetry of form of the rib, by integrating between $\theta=\theta_0$ (at O) and $\theta=\frac{\pi}{2}$ (at highest point, or "crown," of the rib) and doubling the result; as a simple means of obtaining the summation for all the ds's of the rib axis.

Expanding $dU/dM_0=0$, we have (neglecting work of shear),

$$\frac{dU}{dM_0}=2\left[\frac{1}{EI}\int_{\theta_0}^{\pi/2} M\frac{dM}{dM_0}ds+\frac{1}{FE}\int_{\theta_0}^{\pi/2} T\frac{dT}{dM_0}ds\right]=0. \qquad (204)$$

But $dT/dM_0=0$ and the second integral drops out. Substituting from eqs. (200), etc., we derive

$$2\int_{\theta_0}^{\pi/2}[M_0-Hr(\sin\theta-\sin\theta_0)d\theta=0,$$

i.e.,
$$M_0\left(\frac{\pi}{2}-\theta_0\right)-Hr\left[\cos\theta_0-\left(\frac{\pi}{2}-\theta_0\right)\sin\theta_0\right]=0, \quad . \quad (205)$$

or, putting $\frac{\pi}{2}-\theta_0=\theta_1$,

$$\theta_1 M_0-Hr(\cos\theta_0-\theta_1\sin\theta_0)=0. \quad . \quad . \quad . \quad \mathbf{(206)}$$

Again, expanding $dU/dH=\lambda_0$, $=\eta(t-t_0)\,2r\cos\theta_0$, we find, putting $ds=rd\theta$, and neglecting work of shear,

$$2\left[\frac{1}{EI}\int_{\theta_0}^{\pi/2} M\frac{dM}{dH}ds+\frac{1}{FE}\int_{\theta_0}^{\pi/2} T\frac{dT}{dH}ds\right]=\lambda_0;$$

i.e.,

$$\frac{1}{EI}\int_{\theta_0}^{\pi/2}[M_0-Hr(\sin\theta-\sin\theta_0)][-r(\sin\theta-\sin\theta_0)]rd\theta$$
$$+\frac{r}{FE}\int_{\theta_0}^{\pi/2} H(\sin\theta)(\sin\theta)d\theta=\tfrac{1}{2}\lambda_0, \quad . \quad . \quad \mathbf{(207)}$$

whence, after integration and reduction, putting $\frac{\pi}{2}-\theta_0=\theta_1$,

$$[\theta_1\sin\theta_0-\cos\theta_0]M_0+[\theta_1\sin^2\theta_0-\sin 2\theta_0]Hr$$
$$+\tfrac{1}{2}\left(1+\frac{k^2}{r^2}\right)[\theta_1+\tfrac{1}{2}\sin 2\theta_0]Hr=\frac{EI\lambda_0}{2r^2}. \quad . \quad \mathbf{(208)}$$

Elimination of M_0 from (206) and (208) gives

$$H=\frac{\frac{EI}{r^2}\cdot\eta(t-t_0)\cos\theta_0}{\frac{1}{2}\left(1+\frac{k^2}{r^2}\right)(\theta_1+\frac{1}{2}\sin 2\theta_0)-\frac{\cos^2\theta}{\theta_1}} \quad . \quad . \quad (209)$$

With H known, M_0 is found from eq. (206), i.e.,

$$M_0=\left[\frac{\cos\theta_0}{\theta_1}-\sin\theta_0\right]Hr. \quad . \quad . \quad . \quad . \quad (210)$$

If the work of *thrust* were neglected (besides that of shear) the resulting value of H would be obtained from eq. (209) by writing zero in place of the ratio k^2/r^2 in the *denominator* (only).

72. Simpson's Rule for Approximate Integration. In applying the methods of internal work it frequently happens that summations must be made which on a strict mathematical basis are either impossible or extremely cumbersome. In such

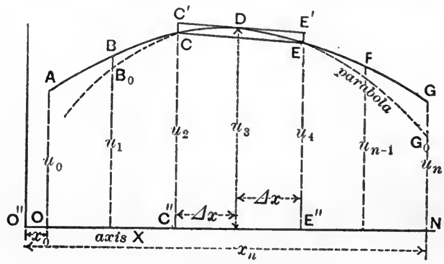

FIG. 42.

cases recourse may be had to **Simpson's Rule,** or other method of approximate integration, as will be illustrated in the next paragraph (§ 73). The present paragraph will consist of a reproduction of the statement and proof of Simpson's Rule taken from the author's Notes and Examples in "Mechanics" (p. 13); as follows:

If *ABCDEFG*, Fig. 42, is a smooth curve and ordinates be drawn from its extremities A and G to the axis X, an approximation to the value of the area so inclosed, $A \ldots D \ldots G \ldots N \ldots O \ldots A$, between the curve and the axis X, is obtained by *Simpson's Rule*, now to be demonstrated. Divide the base ON into an *even* number, n, of *equal* parts, each $=\Delta x$ (so that

$\overline{ON}=n \,.\, \Delta x$), and draw an ordinate from each point of division to the curve, the lengths of these ordinates being u_0, u_1, u_2, etc.; see figure.

Consider the strips of area so formed in consecutive pairs; for example, $CDEE''C''$ is the second pair in this figure (counting from left to right). Conceive a parabola, *with its axis vertical,* to be passed through the points C, D, and E. It will coincide with the real curve between C and E much more closely than would the straight chords CD and DE; and the segment CDE, considered as the segment of this parabola, has an area equal to two-thirds of that of the circumscribing parallelogram $CC'E'E$. Hence, since the area of this pair of strips = *trapezoid* $CEE''C''$ + *parabolic segment* CDE, we may put

$$\left.\begin{array}{c}\textit{Area of pair of}\\ \textit{strips } CE''\end{array}\right\} = 2\Delta x[\tfrac{1}{2}(u_2+u_4)+\tfrac{2}{3}(u_3-\tfrac{1}{2}[u_2+u_4])];$$

which reduces to $\quad \ldots \tfrac{1}{3}\Delta x[u_2+4u_3+u_4].$

Treating all the $\frac{n}{2}$ pairs of strips in a similar manner, we have finally, after writing $\Delta x=(x_n-x_0)\div n$,

$$\left.\begin{array}{c}\textit{Whole area } AG''\\ \textit{(approx.)}\end{array}\right\} = \frac{x_n-x_0}{3n}[u_0+4u_1+2u_2+4u_3+2u_4+\ldots+u_n].$$

The approximation is closer the more numerous the strips and the more accurate the measurement of the ordinates u_0, u_1, u_2, etc.

If the subdivision on the axis X were "infinitely small," an exact value for the area would be expressed by the calculus form $\int_{x=x_0}^{x=x_n} u\,dx$. Hence for any integral of this form, $\int_{x=x_0}^{x=x_n} u\,dx$, if we are only able to determine the particular values (u_0, u_1, etc.) of the variable u corresponding respectively to the abscissæ x_0, $x_0+\Delta x$, $x_0+2\Delta x$, etc. (where $\Delta x=(x_n-x_0)\div n$, n being an *even* number), we can obtain an approximate value of the integral or summation by writing

$$\int_{x_0}^{x_n} u\,dx = \frac{x_n-x_0}{3n}[u_0+4(u_1+u_3+\ldots+u_{n-1}) + 2(u_2+u_4+\ldots+u_{n-2})+u_n]. \quad . \quad . \quad (211)$$

As to the meaning of n, note that the first ordinate on the left is not u_1, but u_0; also that while there are n strips, the number of points of division is $n+1$, counting the extremities O and N.

73. Davit of Variable Cross-section. Circular Quadrant. Deflection of Extremity. Use of Simpson's Rule. In Fig. 43 is shown a davit, $0 \ldots 6$, of wrought iron; its axis being a quarter circle. All sections circular, but of varying radii; smaller toward the free extremity (point 6). Radius of circular axis is r, $=0 \ldots C$, $=50$ in.

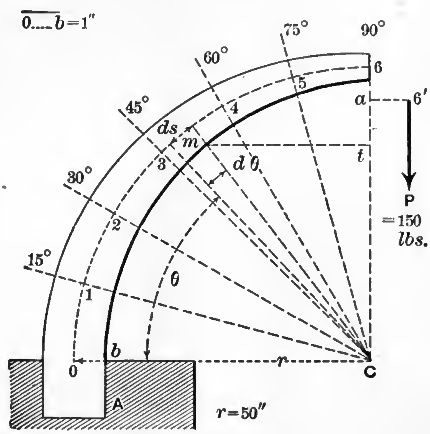

FIG. 43.

At 0 the extremity is "built in," with tangent vertical, in a fixed support. At the other end, 6, a vertical load P, $=150$ lbs., is applied; under whose action the extremity 6 (of the axis of davit) is displaced from 6 to 6′. It is required to determine the vertical projection $6 \ldots a$ of this displacement, neglecting the weight of the davit itself; the value of E being 28,000,000 lbs./in.^{2}. Let us take seven different points 0, 1, 2, etc., equally spaced along the axis from 0 to 6. The angles (θ) of the radii drawn to these points from center C are respectively 0°, 15°, 30°, 45°, 60°, 75°, and 90°; from the horizontal $0 \ldots C$. The radii of the cross-sections (circular) at these points are respectively $r_0=1.0$ in.; $r_1=0.98$; $r_2=0.92$; $r_3=0.86$; $r_4=0.75$; $r_5=0.62$; and $r_6=0.50$ in.; while the corresponding moments of inertia are $I_0=\frac{1}{4}\pi r_0^4=0.7854$ in.^{4}; $I_1=0.7251$ in.^{4};

$I_2=0.5692$; $I_3=0.4298$; $I_4=0.2487$; $I_5=0.1162$: and $I_6=0.0491$ in.^{4}.

As will be seen later, values of $\cos^2\theta$ will also be needed for the seven angles: $\theta_0=0°$, $\theta_1=15°$, $\theta_2=30°$, etc., up to $\theta_6=90°$. These values of $\cos^2\theta$ are respectively, 1.0, 0.9330, 0.7500, 0.5, 0.25, 0.0670, and 0.

Since the vertical distance $6\ldots a$ is the projection upon the line of the external force of the displacement $6\ldots 6'$ we may obtain its value by Castigliano's Theorem [§ 21] by writing

$$\overline{6\ldots a},\ =y,\ =\frac{dU}{dP}, \quad \ldots \ldots \quad (212)$$

i.e. [see § 36],

$$y=\frac{1}{E}\int_0^6 \frac{M}{I}\cdot\frac{dM}{dP}ds; \quad \ldots \ldots \quad (213)$$

(neglecting the internal work *both of thrust and shear*, the radial thickness of the curved beam being small compared with the radius of the circular axis).

From the free body $m\ldots 6$, a radial section having been made at m, we find by moments about m,

$$M=Pr\cos\theta; \quad \text{and} \quad \frac{dM}{dP}=r\cos\theta. \quad \ldots \quad (214)$$

These being substituted in eq. (213), with $ds=rd\theta$, we obtain

$$y=\frac{Pr^3}{E}\int_{\theta=0}^{\theta=90°}\left(\frac{\cos^2\theta}{I}\right)d\theta. \quad \ldots \ldots \quad (215)$$

The integral contains two variables, θ and I. If I were a given algebraic function of θ, this expression could be substituted and the integration performed (provided only known integral forms were encountered). But in the present case, although the curved beam has a smooth taper from 0 to the end 6, and all sections are circles, I is not a known algebraic function of θ. However, the value of I may be computed from the measured radius of section for any assigned value of θ. Simpson's Rule, therefore, offers a means of securing a result accurate enough for practical purposes.

Comparing the integral in eq. (215) with that in the left-hand member of (211), we note that $d\theta$ corresponds to dx; $\frac{\pi}{2}$ to x_n; 0 to x_0; and $\left(\frac{\cos^2\theta}{I}\right)$ to u. Choosing a value of 6 for n, we therefore divide the angle ($=90°$) between point **0** and 6 into six equal parts; with corresponding points on the axis of beam (see Fig. 43) and note that

$$u_0=\frac{\cos^2 0°}{I_0};\quad u_1=\frac{\cos^2 15°}{I_1};\quad u_2=\frac{\cos^2 30°}{I_2};\ \text{etc., to}\ u_6=\frac{\cos^2 90°}{I_6}.$$

Hence, eq. (215) becomes

$$y=\frac{Pr^3}{E}\cdot\frac{\frac{\pi}{2}-0}{3\times 6}\left[\frac{1}{I_0}+\frac{4\cos^2 15°}{I_1}+\frac{2\cos^2 30°}{I_2}\right.$$
$$\left.+\frac{4\cos^2 45°}{I_3}+\frac{2\cos^2 60°}{I_4}+\frac{4\cos^2 75°}{I_5}+0\right]; \quad . \quad (216)$$

or, with numerical values, using the inch and pound as units,

$$y=\frac{150\times(50)^3\cdot\frac{\pi}{2}}{28000000\times 18}\left[\frac{1}{0.7854}+\frac{4\times 0.9330}{0.7251}+\frac{2\times 0.7500}{0.5692}\right.$$
$$\left.+\frac{4\times 0.5000}{0.4298}+\frac{2\times 0.2500}{0.2487}+\frac{4\times 0.0670}{0.1162}\right]=1.05\ \text{in.}$$

as the value of the vertical deflection $\overline{6\ldots a}$.

Similarly, we may determine the horizontal deflection $a\ldots 6'$, using the same method as in § 64 (involving a "dummy" horizontal force at point 6).

(It is here supposed that the elastic limit is not passed in the outer fiber of any section of the curved beam. Let the student test this point).

APPENDIX

Integral Forms. A few integral forms, useful in the mathematical work of this book, are here presented.

$$\int x^n dx = \frac{x^{n+1}}{n+1};$$

$$\int x^{-1} dx, \quad \text{or} \quad \int \frac{dx}{x}, \ = \log_e x;$$

$$\int \sin\theta \, . \, d\theta = -\cos\theta;$$

$$\int \cos\theta \, . \, d\theta = +\sin\theta;$$

$$\int \sin\theta \, . \, \cos\theta \, . \, d\theta = \tfrac{1}{2}\sin^2\theta;$$

$$\int \theta \, . \, \cos\theta. \, d\theta = \theta \, . \, \sin\theta + \cos\theta;$$

$$\int \sin^2\theta \, . \, d\theta = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta;$$

$$\int \cos^2\theta \, . \, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta.$$

Note C. (See foot of p. 30.) If a curve be plotted from the expression for U, as function of T_1, in the seventh line from top of p. 30, with U as (vertical) ordinate and T_1 as (horizontal) abscissa, this curve is evidently the *common parabola;* and, moreover, one with its *geometrical axis vertical;* since it must be a conic section, the highest power of either co-ordinate being the second, and since it is impossible to get more than one value of U for any assigned value of T_1. These characteristics

show that the curve cannot be an ellipse, nor an hyperbola, nor a parabola with its axis other than vertical.

For the angle α, which a tangent line at any point of this curve makes with the horizontal, we have $\tan \alpha = \frac{dU}{dT_1}$, and this is given in eq. (45); which is an equation of the first degree as respects T_1, so that only one value of T_1 satisfies the condition $\frac{dU}{dT_1} = 0$. The point of the curve where $\frac{dU}{dT_1} = 0$ is, of course, the vertex of the parabola, where the tangent line is horizontal, and here the ordinate U is either a maximum or a minimum. It will be a *minimum* if the curve on either side of the vertex *rises*, the final value of U being $+$*infinity;* but a *maximum*, if the curve *descends* on either side, to a final ordinate of $-$*infinity.* But the latter value for U is impossible; since, from the nature of the expression for U, each term is positive for all values of T_1. Hence the value of T_1 obtained by putting $\frac{dU}{dT_1} = 0$ corresponds to a *minimum* value of U.

Note D. (See p. 116.) To prove eqs. (199) a little more specifically let us suppose that the rib is at first supported (i.e., held) by the right-hand support at K, only, and under no load; but at the *higher* temperature, t. The point O of the axis would then be found a distance λ_0, horizontally to the left from where it is actually constrained to be, and the plane of the cross-section would be *parallel* to its ultimate position. Let now the system of independent "loads," consisting of H and the forces forming the couple of moment M_0, be gradually applied (beginning with zero values) until they reach their respective ultimate and actual values. The horizontal displacement of point O of the rib will be $\lambda_0, = \eta(t - t_0)2r \cos \theta_0$, giving $\frac{dU}{dH} = \lambda_0$ (p. 20); while the angle through which the plane of section will have turned is zero, whence $\frac{dU}{dM_0} = 0$ (p. 25).

INDEX

Trigonometric Ratios (Natural); including "*arc*," by which is meant "radians," or "π-measure," or "circular measure;" *e.g.*, arc 100° = 1.7453293, = $\frac{100}{180}$ of π.

arc	degree	sin	cosec	tan	cotan	sec	cos		
0	0	0	infin.	0	infin.	1.0000	1.0000	90	1.5708
0.0175	1	0.0175	57.299	0.0175	57.290	1.0001	0.9998	89	1.5533
.0349	2	.0349	28.654	.0349	28.636	1.0006	.9994	88	1.5359
.0524	3	.0523	19.107	.0524	19.081	1.0014	.9986	87	1.5184
.0698	4	.0698	14.336	.0699	14.301	1.0024	.9976	86	1.5010
.0873	5	.0872	11.474	.0875	11.430	1.0038	.9962	85	1.4835
0.1047	6	0.1045	9.5668	0.1051	9.5144	1.0055	0.9945	84	1.4661
.1222	7	.1219	8.2055	.1228	8.1443	1.0075	.9925	83	1.4486
.1396	8	.1392	7.1853	.1405	7.1154	1.0098	.9903	82	1.4312
.1571	9	.1564	6.3925	.1584	6.3138	1.0125	.9877	81	1.4137
.1745	10	.1736	5.7588	.1763	5.6713	1.0154	.9848	80	1.3963
0.1920	11	0.1908	5.2408	0.1944	5.1446	1.0187	0.9816	79	1.3788
.2094	12	.2079	4.8097	.2126	4.7046	1.0223	.9781	78	1.3614
.2269	13	.2250	4.4454	.2309	4.3315	1.0263	.9744	77	1.3439
.2443	14	.2419	4.1336	.2493	4.0108	1.0306	.9703	76	1.3264
.2618	15	.2588	3.8637	.2679	3.7321	1.0353	.9659	75	1.3090
0.2793	16	0.2756	3.6280	0.2867	3.4874	1.0403	0.9613	74	1.2915
.2967	17	.2924	3.4203	.3057	3.2709	1.0457	.9563	73	1.2741
.3142	18	.3090	3.2361	.3249	3.0777	1.0515	.9511	72	1.2566
.3316	19	.3256	3.0716	.3443	2.9042	1.0576	.9455	71	1.2392
.3491	20	.3420	2.9238	.3640	2.7475	1.0642	.9397	70	1.2217
0.3665	21	0.3584	2.7904	0.3839	2.6051	1.0712	0.9336	69	1.2043
.3840	22	.3746	2.6695	.4040	2.4751	1.0785	.9272	68	1.1868
.4014	23	.3907	2.5593	.4245	2.3559	1.0864	.9205	67	1.1694
.4189	24	.4067	2.4586	.4452	2.2460	1.0946	.9135	66	1.1519
.4363	25	.4226	2.3662	.4663	2.1445	1.1034	.9063	65	1.1345
0.4538	26	0.4384	2.2812	0.4877	2.0503	1.1126	0.8988	64	1.1170
.4712	27	.4540	2.2027	.5095	1.9626	1.1223	.8910	63	1.0996
.4887	28	.4695	2.1301	.5317	1.8807	1.1326	.8829	62	1.0821
.5061	29	.4848	2.0627	.5543	1.8040	1.1434	.8746	61	1.0646
.5236	30	.5000	2.0000	.5774	1.7321	1.1547	.8660	60	1.0472
0.5411	31	0.5150	1.9416	0.6009	1.6643	1.1666	0.8572	59	1.0297
.5585	32	.5299	1.8871	.6249	1.6003	1.1792	.8480	58	1.0123
.5760	33	.5446	1.8361	.6494	1.5399	1.1924	.8387	57	0.9948
.5934	34	.5592	1.7883	.6745	1.4826	1.2062	.8290	56	0.9774
.6109	35	.5736	1.7435	.7002	1.4281	1.2208	.8192	55	0.9599
0.6283	36	0.5878	1.7013	0.7265	1.3764	1.2361	0.8090	54	0.9425
.6458	37	.6018	1.6616	.7536	1.3270	1.2521	.7986	53	0.9250
.6632	38	.6157	1.6243	.7813	1.2799	1.2690	.7880	52	0.9076
.6807	39	.6293	1.5890	.8098	1.2349	1.2868	.7771	51	0.8901
.6981	40	.6428	1.5557	.8391	1.1918	1.3054	.7660	50	0.8727
0.7156	41	0.6561	1.5243	0.8693	1.1504	1.3250	0.7547	49	0.8552
7330	42	.6691	1.4945	.9004	1.1106	1.3456	.7431	48	0.8378
7505	43	.6820	1.4663	.9325	1.0724	1.3673	.7314	47	0.8203
.7679	44	.6947	1.4396	.9657	1 0355	1.3902	.7193	46	0.8028
.7854	45	.7071	1.4142	1.0000	1 0000	1.4142	.7071	45	0.7854
		cos	sec	cotan	tan	cosec	sin	degree	arc

pes net x